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Abelian Forcing Sets

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Many readers of the MONTHLY have encountered particular cases of the following question. Suppose G is a group and n is an integer with the property that $(ab)^n = a^n b^n$ for all a and b in G. Which values of n imply that G is Abelian? Indeed, standard exercises in undergraduate abstract algebra textbooks ([1], [2], [3], [4]) are to show that $n = 2$ and $n = -1$ are two such values. Are there others? If $n \in \mathbb{Z}$, we say that a group G is *n-Abelian* if $(xy)^n = x^n y^n$ for all $x, y \in G$. Thus our question may be reformulated as "for which integers n is an n-Abelian group necessarily abelian?" If p is any prime, consider the non-Abelian group

$$G_p = \left\{ \left(\begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{array} \right) \middle| \, a, b, c \in \mathbb{F}_p \right\}.$$

If p is odd, then $x^p = e$ for all $x \in G_p$. We say that a group G *has exponent* n if $x^n = e$ for all $x \in G$. Thus, G_p has exponent p. Also, G_2 (which is isomorphic to the group of symmetries of a square) has exponent 4. Note that if G is a group with exponent n, then for any integer k, G is kn-Abelian and $(kn+1)$-Abelian. The examples G_p are now sufficient to show that the only integers n for which n-Abelian implies Abelian are $n = 2$ and $n = -1$. Indeed, for p odd, G_p is pk-Abelian and $(pk+1)$-Abelian for any integer k, while G_2 is $4k$-Abelian and $(4k+1)$-Abelian.

More generally, let us call a set of integers T *Abelian forcing* if whenever G is a group with the property that G is n-Abelian for all n in T, then G is Abelian. So far we have seen that the only singleton Abelian forcing sets are $\{-1\}$ and $\{2\}$. What about other sets? Both of Herstein's algebra textbooks ([3, p.31] and [4, p.57]) include the exercise that sets containing three consecutive integers are Abelian forcing. Moreover, one of Herstein's books ([4, p.57]) has an exercise that $\{3, 5\}$ is an Abelian forcing set. In contrast, the set $\{3, 7\}$ is not Abelian forcing, as G_3 is both 3-Abelian and 7-Abelian.

What characterizes the Abelian forcing sets? Although we could not find the answer to this precise question in the literature, some of the essential features of our argument below can be gleaned from a paper by F. Levi [5] written in the group-theoretic language of fifty years ago. (Levi investigated the question of when the mapping $a \to a^n$ is a group endomorphism.) Our formulation of the question, the answer and the proof make the material more accessible to undergraduates.

Theorem. A set T of integers is Abelian forcing if and only if the greatest common divisor of the integers $n(n-1)$ as n ranges over T is 2. (Note that each $n(n-1)$ is even.)

Proof. The necessity of the condition again follows from the examples G_p. For p prime, let $T_p = \{n \in \mathbb{Z} | 2p \text{ divides } n(n-1)\}$. Then for p odd, $T_p = \{pk, pk+1 | k \in \mathbb{Z}\}$, while $T_2 = \{4k, 4k+1 | k \in \mathbb{Z}\}$. From our earlier observation, G_p is n-Abelian for each $n \in T_p$, so T_p is not Abelian forcing. This proves necessity.

To prove sufficiency of the condition, suppose that $T \subseteq \mathbb{Z}$ satisfies $\gcd(n(n-1) | n \in T) = 2$, and G is a group which is n-Abelian for all $n \in T$. Let $S = \{n \in \mathbb{Z} | G \text{ is } n\text{-Abelian}\}$, so that $T \subseteq S$. First note that if $m, n \in S$, then $mn \in S$. Also, if $n \in S$, then for any $x, y \in G$, we have $(xy)^n = x^n y^n$, so that $(yx)^{n-1} = x^{n-1} y^{n-1}$, whence $(yx)^{1-n} = y^{1-n} x^{1-n}$. Thus, if $n \in S$, then $1 - n \in S$. Since $n = 1 - (1 - n)$, the converse holds as well.

Our main difficulty at this point is that S is not closed under addition. However, suppose that $n \in S$ has the property that $x^n \in Z(G)$ (the center of G) for all $x \in G$. Then, for arbitrary $m \in S, x, y \in G$, we have $(xy)^{m+n} = (xy)^m (xy)^n = x^m y^m x^n y^n = x^m x^n y^m y^n = x^{m+n} y^{m+n}$, so $m + n \in S$.

This motivates the definition $R = \{n \in S | x^n \in Z(G) \text{ for all } x \in G\}$. It is easy to see that $n \in R$ if and only if $-n \in R$. Thus, from our previous remark, R is an additive subgroup of $\mathbb{Z}$. We now claim that if $n \in S$, then $n(n-1) \in R$. We do this in several steps.

Note that if $n \in S$, then $1 - n \in S$, so $n(1-n) \in S$. For arbitrary $x, y \in G$, we have $yx^n y^n y^{-1} = y(xy)^n y^{-1} = (yx)^n = y^n x^n$, so that $y^{1-n} x^n = x^n y^{1-n}$. Thus n-th powers commute with $(1-n)$-th powers. Now, for any $x \in G$, $x^{n(1-n)}$ is both an n-th power and a $(1-n)$-th power. Thus, for any $y \in G$, $x^{n(1-n)}$

commutes with both y^n and y^{1-n}, and therefore also with y. This shows that $x^{n(1-n)} \in Z(G)$, so that $n(1-n)$ and thus also $n(n-1)$ are in R.

We are now in position to prove sufficiency. Since that greatest common divisor of the numbers $n(n-1)$ for $n \in T$ is 2, the additive subgroup R of $\mathbb{Z}$ contains 2. Therefore, G is 2-Abelian, and thus Abelian. This proves sufficiency.

Finally, to see that $\{n, n+1, n+2\}$ is Abelian forcing, note that $n(n-1) - 2(n+1)n + (n+2)(n+1) = 2$, so that $\gcd(n(n-1), (n+1)n, (n+2)(n+1)) = 2$.

References

1. J.B. Fraleigh, A First Course in Abstract Algebra, 4th ed., Addison-Wesley, Reading, MA, 1989.

2. J.A. Gallian, Contemporary Abstract Algebra, 2nd ed., D.C. Heath, Lexington, MA, 1990.

3. I.N. Herstein, Topics in Algebra, 2nd ed., John Wiley & Sons, New York, 1975. MR 50 #9456

4. I.N. Herstein, Abstract Algebra, 2nd ed., Macmillan, New York, 1990. MR 92m:00003

5. F.W. Levi, Notes on group theory. I, II. J. Indian Math. Soc. (N.S.), **8** (1944) 1-9. MR 6,40d