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Labeling Grids

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Abstract. We investigate the existence of α-valuations and sequential labelings for a variety of grids in the plane, on a cylinder and on a torus.

1. Introduction. A connected graph with v vertices and e edges is called *graceful* if it is possible to label the vertices x with distinct integers $f(x)$ in $\{0, 1, 2, \ldots, e\}$ so that when each edge, xy, is labeled $|f(x) - f(y)|$, the resulting edge labels are distinct (and thus form the entire set $\{1, 2, 3, \ldots, e\}$).

A graceful labeling f is called an *α-valuation* if there is an integer k such that for any edge xy, either $f(x) \le k < f(y)$ or $f(y) \le k < f(x)$. Since some edge xy is labeled $|f(x) - f(y)| = 1$, the integer k is uniquely determined by the α-valuation, and is said to *characterize* the labeling.

Also, if f is an α-valuation of the graph G, characterized by k, then note that G is bipartite on the vertex sets $X = \{x \in V \mid f(x) > k\}$ and $Y = \{x \in V \mid f(x) \le k\}$. This notation will be used throughout the paper.

Graham and Sloane [7] call a connected graph with v vertices and e edges *harmonious* if its vertices x may be labeled with distinct elements $f(x)$ of $\mathbf{Z}/e\mathbf{Z}$ so that when edge xy is labeled $f(x) + f(y)$, the resulting edge labels are distinct and thus form the entire set $\mathbf{Z}/e\mathbf{Z}$. (If the graph is a tree, i.e. $v = e + 1$, then one vertex label may be repeated.)

Grace [5,6] calls a connected graph with v vertices and e edges *sequential* if its vertices x may be labeled with distinct integers $f(x)$ in $\{0, 1, \ldots, e-1\}$ so that when edge xy is labeled $f(x) + f(y)$, the resulting edge labels form a block of e consecutive integers. If the graph is a tree, then e may be used as a vertex label. Any sequential labeling induces a harmonious labeling by reducing the labels modulo e. Although it seems likely that there are harmonious graphs which are not sequential, there is no known example of such a graph.

In this paper, we find α-valuations and sequential labelings for a variety of graphs of the form $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$, where P_n is a path on n vertices, and C_n is a cycle on n vertices ($n > 2$). Such graphs can be represented as grids in the plane, on a cylinder, or on a torus. Table 1 summarizes our results. Note that all negative results for α-valuations follow simply because these graphs are not bipartite.

Table 1: Summary of Results

Graph	α-valuation	sequential
$P_{2m} \times P_{2n}$	Yes	Yes*
$P_{2m} \times P_{2n+1}$	Yes	Yes
$P_{2m+1} \times P_{2n+1}$	Yes	Yes
$C_{4m} \times P_{2n}$	Yes	Yes*
$C_{4m} \times P_{2n+1}$	Yes	Yes
$C_{4m+2} \times P_{2n}$	Yes	Yes
$C_{4m+2} \times P_{2n+1}$		
$C_{2m+1} \times P_n$	No	Yes
$C_{4m} \times C_{4n}$	Yes	†
$C_{4m} \times C_{4n+2}$	Yes	†
$C_{4m} \times C_{2n+1}$	No	
$C_{4m+2} \times C_{4n+2}$		
$C_{4m+2} \times C_{2n+1}$	No	No
$C_{2m+1} \times C_{2n+1}$	No	No

* except for $(m, n) = (1, 1)$

† $C_4 \times C_{2n}$ is sequential for $n > 1$

Some work with grids has already been done. Graham and Sloane [7] proved that all $C_{2m+1} \times P_n$ are harmonious and Grace [5] has shown these graphs are also sequential.

Maheo [9] has shown that the graphs $P_2 \times P_n$ and $C_4 \times P_n$ have α-valuations, Graham and Sloane [7] have shown that $P_2 \times P_n$ are harmonious, Frucht and Gallian [1] have shown that $P_2 \times C_{2n}$ have α-valuations and $P_2 \times C_{2n+1}$ are graceful, Gallian, Prout and Winters [4] have shown that $P_2 \times C_{2n}$ are sequential for $n > 2$, and Huang and Skiena [8] have shown that $C_{4m+2} \times P_{2n+1}$ are graceful.

In his survey paper on graph labelings, Gallian [2] opined that among the open problems on graph labeling he considered those involving grids to be the most attractive. Gallian again singled out these problems for attention in the Unsolved Problems section of the *American Mathematical Monthly* [3].

2. Relationship between α-valuations and sequential labelings. In [5,6], Grace uses the following construction to prove the corollary to this useful theorem.

Theorem 2.1. If the graph G, with e edges, has an α-valuation, then the vertices x of G can be labeled with distinct integers $f(x)$ in $\{0, 1, 2, \ldots, e\}$ so that the set of induced labels on edges xy, $\{f(x) + f(y)\}$ is a block of e consecutive integers.

Proof. Suppose g is an α-valuation of G, characterized by k. Recall that G is bipartite on $X = \{x \mid g(x) > k\}$ and $Y = \{x \mid g(x) \leq k\}$. Define f by

$$f(x) = \begin{cases} g(x) & \text{if } x \in X \\ k - g(x) & \text{if } x \in Y \end{cases}$$

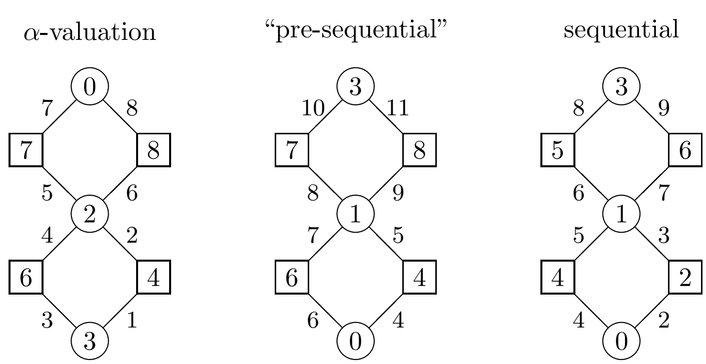
Note that f is injective. Also, if xy is an edge, $x \in X, y \in Y$, then

$$f(x) + f(y) = g(x) + k - g(y) = k + |g(x) - g(y)|\,,$$

so the set of edge labels is $\{k+1, k+2, \ldots, k+e\}$ as desired. $\square$

Corollary 2.2. If a tree has an α-valuation, then it is sequential.

2.3. Bipartite label-shifting. Let G be a graph that is bipartite on X and Y, and suppose we have some labeling, f, of the vertices of G, that induces the labels $f(x) + f(y)$ on the edges xy. If we define a new labeling by incrementing the labels of vertices in X by a and incrementing the labels of vertices in Y by b, the induced edge labels have all been incremented by $a + b$. Thus, this type of change *does not affect* whether or not the edge labels are distinct consecutive integers (or if they form a complete residue system modulo e), but it *may well affect* whether or not the vertex labels are distinct and in the desired range.



This construction, when used together with Grace's construction, can be a powerful tool. For example, start with an α-valuation of $C_4^{(2)}$, the one point union of two copies of C_4, apply Grace's construction, and then subtract 2 from each boxed vertex to get a sequential labeling, as above.

3. Planar Grids. We begin with a simple remark that has far-reaching applications. Suppose the graph G has an α-valuation characterized by k. Recall G is then bipartite on the vertex sets X and Y, as defined in Section 1. If we add x to the labels of vertices in X, and y to the labels of vertices in Y, then the edge labels become the integers from $x-y+1$ through $x-y+e$ (if $x \geq y$).

Theorem 3.1. The graph $P_m \times P_n$ has an α-valuation.

Proof. Let J denote the α-valuation of P_m

$$0, m-1, 1, m-2, 2, m-3, \ldots$$

and let $R(x)$ be the labeling obtained from J by adding $(2mn-2m-n+1)-x$ to the higher labels, x to the lower labels. Let $K=(m-1)-J$ denote the α-valuation of P_m

$$m-1, 0, m-2, 1, m-3, 2, \ldots$$

and let $S(x)$ be the labeling obtained from K by adding $(2mn-2m-n+2)-x$ to the higher labels, and x to the lower labels.

Now, consider $P_m \times P_n$ to be n juxtaposed copies of P_m. Label the $(2k+1)^{st}$ copy of P_m with $R(k(2m-1))$, the $2k^{th}$ copy with $S(k(2m-1)+(1-m))$.

The edge labels of the r^{th} copy of P_m are the integers from $(2mn-n+1)-r(2m-1)$ through $(2mn-n+1)-r(2m-1)+(m-2)$, and the labels of the edges joining the r^{th} and $(r+1)^{st}$ copies of P_m are the integers from $(2mn-n+1)-(r+1)(2m-1)+(m-1)$ through $(2mn-n+1)-r(2m-1)-1$. It follows that this is a graceful labeling, and inspection shows that it is also an α-valuation. $\square$

This labeling is particularly nice, as the edges are labeled consecutively. See labelings of $P_4 \times P_5$ and $P_4 \times P_6$ below:

α-valuation of $P_4 \times P_5$

0	28	7	21	14
31	4	24	11	17
1	27	8	20	15
30	5	23	12	16

α-valuation of $P_4 \times P_6$

0	35	7	28	14	21
38	4	31	11	24	18
1	34	8	27	15	20
37	5	30	12	23	19

Also nice is the following corollary.

Corollary 3.2. The graph $P_{2m} \times P_{2n+1}$ is sequential.

Proof. Apply Grace's construction to the α-valuation given above of $P_{2m} \times P_{2n+1}$, and subtract $2m$ from the higher vertex labels. A routine verification shows that this is indeed a sequential labeling. □

Again, the edges are labeled consecutively; we illustrate for $P_4 \times P_5$ below:

α-valuation of $P_4 \times P_5$

0	28	7	21	14
31	4	24	11	17
1	27	8	20	15
30	5	23	12	16

sequential labeling of $P_4 \times P_5$

15	24	8	17	1
27	11	20	4	13
14	23	7	16	0
26	10	19	3	12

A slight variation will give a sequential labeling of $P_{2m} \times P_{2n}$. However, it is easy to see that $P_2 \times P_2 = C_4$ is not *harmonious*. The following result shows that this is the only exception among these graphs.

Theorem 3.3. If $n > 1$, then $P_{2m} \times P_{2n}$ has a sequential labeling.

Proof. Start with the α-valuation of $P_{2m} \times P_{2n}$ given in 3.1, and subtract 1 from each vertex label of the $2n^{th}$ copy of P_{2m}. This doesn't change any edge labels of the $2n^{th}$ copy of P_{2m}, and permutes adjacent pairs of edges joining the last two copies of P_{2m}. Since no other edge labels are affected, this is also an α-valuation (characterized by a different k).

Now apply Grace's construction, and subtract $2m$ from the higher vertex labels. It is straightforward to check that this is a sequential labeling. □

We illustrate the process for $P_6 \times P_4$:

α-valuation of $P_6 \times P_4$

0	33	11	22
38	6	27	17
1	32	12	21
37	7	26	18
2	31	13	20
36	8	25	19

new α-valuation of $P_6 \times P_4$

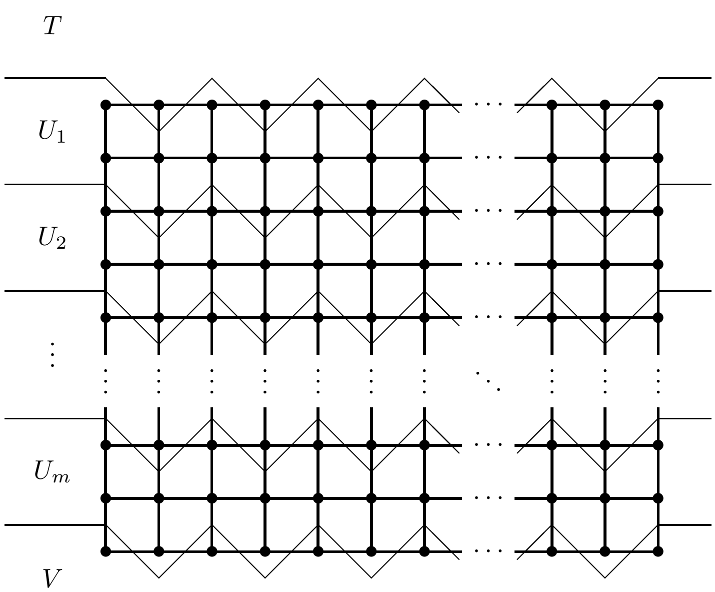
0	33	11	21
38	6	27	16
1	32	12	20
37	7	26	17
2	31	13	19
36	8	25	18

sequential labeling of $P_6 \times P_4$

18	27	7	15
32	12	21	2
17	26	6	14
31	11	20	1
16	25	5	13
30	10	19	0

Theorem 3.4. The graph $P_{2m+1} \times P_{2n+1}$ has a sequential labeling.

Proof. Partition the vertices of $P_{2m+1} \times P_{2n+1}$ into the sets T, U_1, U_2, U_3, $\ldots$, U_m and V as shown.



Label the vertices of T with $x-3n, x-3n+1, x-3n+2, \ldots, x-2n-1$, where $x = 4mn + m + 2n$. Label U_1 as follows

0		2		4			$2n$
x	1	$x+2$	3	$x+4$	$\cdots$	$2n-1$	$x+2n$
	$x+1$		$x+3$			$x+2n-1$	

and label U_i by adding $(i-1)(4n+1)$ to corresponding vertices of U_1. Label the vertices of V with $x-2n, x-2n+1, x-2n+2, \ldots, x-n$.

The edges of the graph are then labeled consecutively in a zigzag pattern, which is illustrated by $P_5 \times P_7$. $\square$

sequential labeling of $P_5 \times P_7$

0	23	2	24	4	25	6
32	1	34	3	36	5	38
13	33	15	35	17	37	19
45	14	47	16	49	18	51
26	46	27	48	28	50	29

Corollary 3.5. All grids $P_m \times P_n$ except $P_2 \times P_2$ are sequential.

4. Cylindrical Grids. Let L represent the α-labeling of C_{4m}: $0, 4m, 1, 4m-1, \ldots, m, 3m-1, m+1, \ldots, 2m$. Note that the label $3m$ is unused. Suppose X and Y are the two vertex-subsets of the bipartite graph C_{4m} (as described above). Define $L(x,y)$ to be the vertex-labeling obtained from L by adding x to each label in X and y to each label in Y. Define $L(x,y;r)$ to be the labeling obtained from $L(x,y)$ by rotating the labeling by r vertices (to the right, i.e. a,b,c,d rotated through 1 vertex yields d,a,b,c). Notice that the edge-labels of $L(x,y;r)$ range from $(1+x-y)$ to $(4m+x-y)$.

Theorem 4.1. The graph $C_{4m} \times P_n$ has an α-valuation.

Proof. Label the n copies of C_{4m} with: $L(8mn-8m, 0; 0)$, $L(8mn-12m, 4m; 1)$, $L(8mn-16m, 8m; 0)$, $\ldots$, $L(4mn-4m, 4mn-4m; k)$, where $k=0$ if n is odd, and $k=1$ if n is even. Notice that every second labeling is rotated by 1, and in each case x is decreased by $4m$ while y is increased by $4m$. The i^{th} copy of C_{4m} includes the edges $(8mn-8mi)+1$ through $(8mn-8mi)+4m$. The edges that join the i^{th} and $(i+1)^{st}$ copies of C_{4m} include the edges $(8mn-8mi)+(4m+1)$ through $(8mn-8mi)+8m$. Thus the edges of the graph cover the integers from 1 to $(8mn-4m)$ as desired. By inspection, the labeling is an α-valuation. $\square$

The labeling of $C_4 \times P_5$ is shown:

α-valuation of $C_4 \times P_5$

0	30	8	22	16
36	4	28	12	20
1	32	9	24	17
34	5	26	13	18

The following result employs an important technique that will be used several times.

Theorem 4.2. The graph $C_{4m} \times P_{2n+1}$ has a sequential labeling.

Proof. Start with the α-valuation given in 3.1 of $P_{2n+1} \times P_{4m}$. Transpose the graph and its labeling, as though it were a matrix. This gives an α-valuation of $P_{4m} \times P_{2n+1}$. Now apply Grace's construction, subtract $8mn + 2m - n$ from the higher labels, and add $8mn + 2m - 3n$ to the lower labels to get a sequential labeling of $P_{4m} \times P_{2n+1}$.

Connect the $4m^{th}$ copy of P_{2n+1} to the first, so our graph becomes $C_{4m} \times P_{2n+1}$. These new edges have the same labels as those edges connecting the $2m^{th}$ and $(2m+1)^{st}$ copy of P_{2n+1}.

Let U denote the first $2m$ copies of P_{2n+1} and the edges between them. The labels of edges in U are those integers from A through B. The labels of edges with one vertex in U are the integers from $A - 2n - 1$ through $A - 1$, each occurring twice. Now add $2n + 1$ to the higher labels in U. (Since $P_{4m} \times P_{2n+1}$ is not a tree, and since we've added $2n + 1$ edges to it, the vertex labels stay in range for a sequential labeling.) The edges contained in U now have labels $A + 2n + 1$ through $B + 2n + 1$, and of the two edges that used to have the same label, one has been increased by $2n + 1$, the other unchanged. Therefore, the edges with one vertex in U now have labels $A - 2n - 1$ through $A + 2n$, and we have a sequential labeling. $\square$

α-valuation $P_4 \times P_5$

0	31	1	30	2
27	5	26	6	25
9	22	10	21	11
18	14	17	15	16

sequential labeling of $P_4 \times P_5$

27	15	26	14	25
11	22	10	21	9
18	6	17	5	16
2	13	1	12	0

sequential labeling of $C_4 \times P_5$

32	15	31	14	30
11	27	10	26	9
18	6	17	5	16
2	13	1	12	0

The same technique is used for the following result.

Theorem 4.3. If $m > 1$, then $C_{4m} \times P_{2n}$ is sequential.

Proof. Start with the α-valuation of $P_{2n} \times P_{4m}$ given in 3.1, and transpose it, as in 4.2, to get an α-valuation of $P_{4m} \times P_{2n}$. Add 1 to each label of the $(4m-1)^{st}$ copy of P_{2n}. Similar to the proof of 3.3, the only affected edge labels are those joining the $(4m-2)^{nd}$ and $(4m-1)^{st}$ copies of P_{2n}, and those joining the $(4m-1)^{st}$ and $4m^{th}$ copies. Furthermore, these labels are permuted in adjacent pairs, so this is still an α-valuation.

Now apply Grace's construction, and subtract $2n$ from the higher labels to get a sequential labeling. Join the first copy of P_{2n} to the last, to get $C_{4m} \times P_{2n}$, and add $2n$ to the higher labels of the first $2m$ copies of P_{2n}. As before, this gives a sequential labeling of $C_{4m} \times P_{2n}$. $\square$

α-valuation of $P_8 \times P_4$

0	52	1	51
49	4	48	5
7	45	8	44
42	11	41	12
14	38	15	37
35	18	34	19
21	31	22	30
28	25	27	26

new α-valuation of $P_8 \times P_4$

0	52	1	51
49	4	48	5
7	45	8	44
42	11	41	12
14	38	15	37
35	18	34	19
22	32	23	31
28	25	27	26

sequential labeling of $P_8 \times P_4$

26	48	25	47
45	22	44	21
19	41	18	40
38	15	37	14
12	34	11	33
31	8	30	7
4	28	3	27
24	1	23	0

sequential labeling of $C_8 \times P_4$

26	52	25	51
49	22	48	21
19	45	18	44
42	15	41	14
12	34	11	33
31	8	30	7
4	28	3	27
24	1	23	0

In light of Grace's construction, it is not surprising to find that this technique also comes in handy in the realm of α-valuations.

Theorem 4.4. The graph $C_{4m+2} \times P_{2n}$ has an α-valuation.

Proof. Transpose the α-valuation of $P_{2n} \times P_{4m+2}$ given in 3.1, to get an α-valuation of $P_{4m+2} \times P_{2n}$. Now subtract 1 from each label of the $(4m+2)^{nd}$ copy of P_{2n}. We've seen earlier that this is also an α-valuation. Now connect the first copy of P_{2n} to the last, to get $C_{4m+2} \times P_{2n}$, and add $2n$ to the higher labels of the first $2m+1$ copies of P_{2n}. Similar to before, this gives an α-valuation of $C_{4m+2} \times P_{2n}$. □

We also have the following result.

Corollary 4.5. The graph $C_{4m+2} \times P_{2n}$ is sequential.

Proof. Apply Grace's construction to the α-valuation given above, and subtract $2n$ from the higher labels to get a sequential labeling. □

We illustrate for $C_6 \times P_4$:

α-valuation of $P_6 \times P_4$

0	38	1	37
35	4	34	5
7	31	8	30
28	11	27	12
14	24	15	23
21	18	20	19

new α-valuation of $P_6 \times P_4$

0	38	1	37
35	4	34	5
7	31	8	30
28	11	27	12
14	24	15	23
20	17	19	18

α-valuation of $C_6 \times P_4$

0	42	1	41
39	4	38	5
7	35	8	34
28	11	27	12
14	24	15	23
20	17	19	18

sequential labeling of $C_6 \times P_4$

18	38	17	37
35	14	34	13
11	31	10	30
24	7	23	6
4	20	3	19
16	1	15	0

Theorems 4.2 and 4.3 show that all graphs $C_{4n} \times P_m$ are sequential, except for $C_4 \times P_{2m}$. In fact, Graham and Sloane [7] used an exhaustive

computer search to show that $C_4 \times P_2$ is not even harmonious. However, our next result shows that this is the sole exception among these graphs.

Theorem 4.6. If $n > 2$, then $C_4 \times P_n$ is sequential.

Proof. Let M denote the labeling of C_4 by 3, $4n-1$, 5, $4n$, and N the labeling 0, $4n-5$, 2, $4n-6$. Let $M(x)$ denote the labeling obtained from M by adding x to the vertex labels, and let $M(x;t)$ be the labeling obtained by rotating $M(x)$ through t vertices (as in the proof of Theorem 4.1). Define $N(x;t)$ similarly.

Now, label the first copy of C_4 $N(0;0)$, the last $N(4(n-1);n+1)$. Label the r^{th} copy $(1 < r < n)$ $M(4(r-2);r-1)$.

The edge labels on the r^{th} copy of C_4 are $8r+4n-14$ through $8r+4n-11$, and the edges joining the r^{th} and $(r+1)^{st}$ copies have labels $8r+4n-10$ through $8r+4n-7$. One readily checks that the vertex labels are distinct and in the desired range, so this is a sequential labeling. □

We illustrate for $C_4 \times P_5$ and $C_4 \times P_6$:

sequential labeling of $C_4 \times P_5$

0	20	9	27	18
15	3	24	13	30
2	19	7	28	16
14	5	23	11	31

sequential labeling of $C_4 \times P_6$

0	24	9	31	15	39
19	3	28	13	35	22
2	23	7	32	17	38
18	5	27	11	36	20

Graham and Sloane [7] note that any harmonious labeling of C_{2m+1} extends to a harmonious labeling of $C_{2m+1} \times P_n$. The same is true for sequential labelings, as noted by Grace [5]. Indeed, if Q is any sequential labeling of C_{2m+1}, (for example, 0, $m+1$, 1, $m+2$, 2, …, $m-1$, $2m$, m), and $Q(x;t)$ defined as in 4.6, then $Q(0;0)$, $Q(2m+1;1)$, $Q(2(2m+1);0)$, $Q(3(2m+1);1)$, … is a sequential labeling of $C_{2m+1} \times P_n$. Illustrated is $C_5 \times P_6$:

sequential labeling of $C_5 \times P_6$

0	7	10	17	20	27
3	5	13	15	23	25
1	8	11	18	21	28
4	6	14	16	24	26
2	9	12	19	22	29

5. Toroidal Grids. These graphs seem to be more challenging, and we have settled very few cases.

Theorem 5.1. The graph $C_{4m} \times C_{4n}$ has an α-valuation.

Proof. Start with the α-valuation of $C_{4m} \times P_{4n}$ given in 4.1. Connect the first and last copies of C_{4m} to get the graph $C_{4m} \times C_{4n}$. Now, add $4m$ to the higher labels in the first $2n$ copies of C_{4m} to get an α-valuation. □

We illustrate for $C_4 \times C_8$:

α-valuation of $C_4 \times P_8$

0	54	8	46	16	38	24	30
60	4	52	12	44	20	36	28
1	56	9	48	17	40	25	32
58	5	50	13	42	21	34	29

α-valuation of $C_4 \times C_8$

0	58	8	50	16	38	24	30
64	4	56	12	44	20	36	28
1	60	9	52	17	40	25	32
62	5	54	13	42	21	34	29

A slight variation will work for $C_{4m} \times C_{4n+2}$.

Theorem 5.2. The graph $C_{4m} \times C_{4n+2}$ has an α-valuation.

Proof. Start with the α-valuation given in 4.1 of $C_{4m} \times P_{4n+2}$, except label the $(4n+2)^{nd}$ copy of C_{4m} with $L(16mn+4m, 16mn+4m; -1)$ instead of $L(16mn+4m, 16mn+4m; 1)$. As in 4.1, this is also an α-valuation. Now, connect the first and last copies of C_{4m}, to get $C_{4m} \times C_{4n+2}$, and add $4m$ to the higher labels in the first $2n+1$ copies of C_{4m}. This gives an α-valuation of $C_{4m} \times C_{4n+2}$. □

We illustrate for $C_4 \times C_6$:

original α-valuation of $C_4 \times P_6$

0	38	8	30	16	22
44	4	36	12	28	20
1	40	9	32	17	24
42	5	34	13	26	21

new α-valuation of $C_4 \times P_6$

0	38	8	30	16	24
44	4	36	12	28	21
1	40	9	32	17	22
42	5	34	13	26	20

α-valuation of $C_4 \times C_6$

0	42	8	30	16	24
48	4	40	12	28	21
1	44	9	32	17	22
46	5	38	13	26	20

Our only families of sequential toroidal grids come from the sequential labeling of $C_4 \times P_n$ given in 4.6.

Theorem 5.3. The graph $C_4 \times C_{4n}$ is sequential.

Proof. Start with the sequential labeling of $C_4 \times P_{4n}$ given in 4.6. Connect the first copy of C_4 to the last copy, and add 4 to the higher labels of the last $2n$ copies of C_4 to get a sequential labeling of $C_4 \times C_{4n}$. □

sequential labeling of $C_4 \times P_8$

0	32	9	39	15	48	25	54
26	5	35	11	44	21	51	30
2	31	7	40	17	47	23	55
27	3	36	13	43	19	52	28

sequential labeling of $C_4 \times C_8$

0	32	9	39	15	52	25	58
26	5	35	11	48	21	55	30
2	31	7	40	17	51	23	59
27	3	36	13	47	19	56	28

A slight twist will give us a sequential labeling of $C_4 \times C_{4n+2}$.

Theorem 5.4. The graph $C_4 \times C_{4n+2}$ is sequential.

Proof. Start with the sequential labeling of $C_4 \times P_{4n+2}$ given in 4.6, and rotate the labels of the first two copies of C_4 through 2 vertices. It is easily verified that this is also a sequential labeling. Now, join the first and last copies of C_4, and add 4 to the higher labels of the last $2n+1$ copies of C_4. This gives a sequential labeling of $C_4 \times C_{4n+2}$. □

We illustrate for $C_4 \times C_6$:

original sequential labeling of $C_4 \times P_6$

0	24	9	31	15	39
19	3	28	13	35	22
2	23	7	32	17	38
18	5	27	11	36	20

new sequential labeling of $C_4 \times P_6$

2	23	9	31	15	39
18	5	28	13	35	22
0	24	7	32	17	38
19	3	27	11	36	20

sequential labeling of $C_4 \times C_6$

2	23	9	35	15	43
18	5	28	13	39	22
0	24	7	36	17	42
19	3	27	11	40	20

6. Negative results. In our opening remarks, we note that if G has an α-valuation, then G is bipartite. This necessary condition is responsible for the four negative results for α-valuations. There is also a necessary condition for some graphs to be graceful.

Theorem 6.1. Suppose G is any graph with a graceful labeling, and let $e_1, e_2, \ldots, e_n$ be any edge-cycle of G. Then we can divide the edges e_1 through e_n into two subsets such that the sum of edge-labels of one subset equals that of the other subset.

Proof. Let v_i be the vertex shared by e_i and e_{i+1} (where $e_{n+1} = e_1$), and let $|v|$ and $|e|$ be the labels on v and e respectively. Then $|v_2| = |v_1| \pm |e_2|$, $|v_3| = |v_2| \pm |e_3|$, etc. Combining these equations, we get $|v_1| = |v_1| \pm |e_2| \pm |e_3| \pm \cdots \pm |e_1|$, so $0 = \pm |e_1| \pm |e_2| \pm \cdots \pm |e_n|$. Those edges preceded by a plus sign in this equation form one subset; the other edges form the second subset. □

Corollary 6.2. Let G and $e_1, \ldots, e_n$ be as described in Theorem 6.1. Then $\sum_{i=1}^{n} |e_i|$ is an even integer.

The following corollary was first given by Rosa [10], however we provide an alternate proof.

Corollary 6.3. Let G be any graph with a graceful labeling. Suppose G has e edges, and suppose every vertex of G has an even valence. Then either $e \equiv 3 \bmod 4$ or $e \equiv 0 \bmod 4$.

Proof. G has an Eulerian cycle, so by Corollary 6.2, the sum of the edges is even. This sum is $e(e+1)/2$, so $e \equiv 3 \bmod 4$ or $e \equiv 0 \bmod 4$. □

Corollary 6.4. If m and n are odd, then $C_m \times C_n$ is *not* graceful.

The only general necessary condition for a graph to be harmonious is due to Graham and Sloane [7]. We include the proof for completeness.

Theorem 6.5. If the harmonious graph G has an even number e of edges, and for some $k \geq 0$, the degree of each vertex in G is a multiple of 2^k, then e is a multiple of 2^{k+1}.

Proof. Let f be a harmonious labeling, let $e = 2e'$, let $v(x)$ denote the degree of the vertex x and let $v(x) = 2^k v'(x)$. Then we have

$$e' \equiv \sum_{n=1}^{e} n \equiv \sum_{xy \in E(G)} f(xy) = \sum_{xy \in E(G)} f(x) + f(y) =$$

$$\sum_{x \in V(G)} v(x) f(x) = 2^k \sum_{x \in V(G)} v'(x) f(x) \bmod e.$$

Thus, $0 \not\equiv e' \equiv 2^k \sum v'(x)f(x)$, while $0 \equiv 2e' \equiv 2^{k+1} \sum v'(x)f(x)$. Therefore, the element $\sum_{v \in V(G)} v'(x)f(x)$ has order exactly 2^{k+1} in $\mathbf{Z}/e\mathbf{Z}$, whence 2^{k+1} divides e. $\square$

Corollary 6.6. If 4 does not divide mn, then $C_m \times C_n$ is not harmonious (and hence not sequential).

7. Conclusion. It remains unknown whether or not several classes of grids have various labelings (see gaps in Table 1). Since we have been unable to complete Table 1, we pose as an open question the determination of the missing entries. Although $C_4 \times C_3, C_4 \times C_5, C_4 \times C_7, C_4 \times C_9$ and $C_6 \times C_3$ do not have α-valuations, Eric Wepsic has found graceful labelings for them with the aid of a computer. The sequential labeling of $C_3 \times C_4$ shown below was found by the authors by trial and error. Likewise, the graphs $C_{2m+1} \times P_n$ do not have α-valuations but Huang and Skiena [8] have found graceful labelings for them.

$C_3 \times C_4$ is sequential

0	5	10	7
4	9	15	12
6	3	8	14

To this point, numerous papers have been devoted to labeling various types of graphs, yet few results can be easily generalized. For instance Rosa's criterion (see Corollary 6.3) is the only known necessary condition for existence of graceful labelings, and no conditions are known to be sufficient. Of more interest than the completion of Table 1 would be the development of some general theory of labeling graphs.

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