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THE LOCAL STARK CONJECTURE AT A REAL PLACE

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Abstract. A refinement of the rank 1 abelian Stark conjecture has been formulated by B. Gross. This conjecture predicts some $\mathfrak{p}$-adic analytic nature of a modification of the Stark unit. The conjecture makes perfect sense even when $\mathfrak{p}$ is an archimedean place. Here we consider the conjecture when $\mathfrak{p}$ is a real place, and interpret it in terms of 2-adic properties of special values of L-functions. We prove the conjecture for CM extensions; here the original Stark conjecture is uninteresting, but the refined conjecture is non-trivial. In more generality, we show that, under mild hypotheses, if the subgroup of the Galois group generated by complex conjugations has less than full rank, then the refined conjecture implies that the Stark unit should be a square. This phenomenon has been discovered by Dummit and Hayes in a particular type of situation. We show that it should hold in much greater generality.

1. Introduction

Let k be a number field and $\mathcal{O}$ its ring of integers. The *zeta function* for k is defined by an infinite series

$$\zeta_k(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}} (\mathbf{N}\mathfrak{a})^{-s}$$

which converges absolutely on the half plane $\Re(s) > 1$. It is a classical result that the above definition has a meromorphic continuation to the entire complex plane, with only a simple pole at $s = 1$. Moreover, ζ_k has a functional equation that relates $\zeta_k(s)$ and $\zeta_k(1-s)$. Under this functional equation, the formula for the residue at $s = 1$ transforms into a simple form of the leading coefficient of the Taylor series at $s = 0$, namely

$$\zeta_k(s) = -\frac{h_k R_k}{w_k} s^n + O(s^{n+1}) \qquad \text{near } s = 0 \tag{1}$$

where h_k is the class number, R_k is the regulator and w_k is the number of roots of unity in k. Also, $n = r_1 + r_2 - 1$, where r_1 [respectively, r_2] is the number of real [respectively,

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complex] places of k. In particular, n is the rank of the unit group $\mathcal{O}^*$, by Dirichlet's unit theorem.

In a series of papers [11, 12, 13, 15] H. Stark developed a conjecture about the leading coefficient of the Taylor series at $s = 0$ of Artin L-functions. He conjectured that this leading coefficient should have an analogous form, namely that it should be a product of an algebraic number with a transcendental number (the "Stark regulator"). However, this information is subtle, and it is not fully understood, even conjecturally, the precise form this coefficient should have.

Stark's Conjecture

In the case of abelian L-functions, and especially when the order of vanishing is 1, a more precise conjecture is available. In order to state the conjecture, we will introduce some notations and conventions.

Let K/k be an abelian extension of global fields with Galois group G. Let S be a finite set of places of k that contains all the archimedean places and any places that ramify in K. Suppose that $\#S \geq 2$ and that S contains a place, $\mathfrak{p}_0$, that splits completely in K. Also let $\mathfrak{P}_0$ be one of the places of K lying over $\mathfrak{p}_0$. The hypotheses on S imply that for any character, $\chi \in \widehat{G}$, the L-function $L_S(\chi, s)$ vanishes at $s = 0$ (see Proposition 4.1 below). Therefore we are interested in the coefficient of s^1 in the Taylor series of $L_S(\chi, s)$, or equivalently, the value of $L'_S(\chi, 0)$.

Let U_K denote the $S(K)$-units of K, i.e. those elements of K that are units at every place not lying over a place in S. If $\#S \geq 3$, define

$$U^{(\mathfrak{p}_0)} = \left\{ x \in U_K \quad \middle| \quad |x|_{\mathfrak{P}} = 1 \quad \text{for all } \mathfrak{P} \text{ not dividing } \mathfrak{p}_0 \right\},$$

but if $\#S = 2$, say $S = \{\mathfrak{p}_0, \mathfrak{p}_1\}$, then choose a place, $\mathfrak{P}_1$ of K lying over $\mathfrak{p}_1$, and define

$$U^{(\mathfrak{p}_0)} = \left\{ x \in U_K \quad \middle| \quad |x^g|_{\mathfrak{P}_1} = |x|_{\mathfrak{P}_1} \quad \text{for all } g \text{ in } G \right\}.$$

Clearly, this latter definition does not depend upon the choice of $\mathfrak{P}_1$ of K lying over $\mathfrak{p}_1$.

Conjecture 1.1. (The rank 1 abelian Stark conjecture) *With the notation above, there is $\varepsilon \in U^{(\mathfrak{p}_0)}$ such that*

- $L'_S(\chi, 0) = -\dfrac{1}{w_K} \displaystyle\sum_{g \in G} \chi(g) \log |\varepsilon^g|_{\mathfrak{P}_0}$ *for all $\chi \in \widehat{G}$, and*
- $K(\varepsilon^{1/w_K})$ *is abelian over k.*

The quantity ε is called the *Stark unit*. The conditions imposed on it by Conjecture 1.1 determine its absolute value at every place of K. Therefore, ε is uniquely determined, up to a multiple of a root of unity in K, and the truth of the conjecture is independent of the choice of ε. In general however, there seems to be no canonical choice among the w_K possibilities.

Conjecture 1.1 is known to be true when the base field $k = \mathbb{Q}$, where it amounts to classical results of Stickelberger on the factorization of Gauss sums. Stark has also proved it when the base field, k, is a quadratic imaginary field, by using the theory of complex multiplication. These two cases form the foundation of theoretical evidence in support of the conjecture. Conjecture 1.1 has also been verified numerically in many cases, for example, see [5, 14].

The Local Stark Conjecture

Benedict Gross has formulated a conjectural generalization of formula (1) above, where the regulator has been replaced by a determinant in a group ring. The group elements are obtained as local reciprocity maps applied to various units. By analogy, he formulates the "local Stark conjecture", which predicts the value of a local reciprocity map on a modified Stark unit. This can be viewed as an attempt to understand the $\mathfrak{p}$-adic analytic nature of the Stark unit.

Let K/k and S be as above, and let L be an overfield of K that is abelian over k and unramified outside S. Let ε be the hypothetical Stark unit for K/k. As noted above, ε is only determined up to a root of unity in K. Let $\lambda = \varepsilon^{1/w_K}$, and consider the extension $L(\lambda)/k$. This is the compositum of the abelian extension L/k with the (conjecturally) abelian extension $K(\lambda)/k$, and thus is (conjecturally) abelian.

Let $\mathfrak{q}$ be a place of k which is not in S, nor divides the number of roots of unity in $L(\lambda)$, and let $\varphi_\mathfrak{q}$ be its Frobenius element in $\mathrm{Gal}(L(\lambda)/k)$, which makes sense, as this extension is abelian. Now define

$$\varepsilon_\mathfrak{q} = \lambda^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}}.$$

The quantity $\varepsilon_\mathfrak{q}$ is called the *modified Stark unit*. Its definition does not depend upon the choice of the Stark unit, ε, nor on the choice of its w_K-th root, λ. Moreover, we claim that $\varepsilon_\mathfrak{q} \in K$. To see this, let $\tau \in \mathrm{Gal}(L(\lambda)/k)$ be an arbitrary element, and note that $(\lambda^\tau)^{w_K} = (\lambda^{w_K})^\tau = \varepsilon^\tau = \varepsilon = \lambda^{w_K}$. Thus, $\lambda^\tau = \zeta\lambda$, where ζ is some w_K-th root of unity. Now, using the fact that $\mathrm{Gal}(L(\lambda)/k)$ is abelian, we have

$$\varepsilon_\mathfrak{q}^\tau = (\lambda^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}})^\tau = (\lambda^\tau)^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}} = (\zeta\lambda)^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}} = \zeta^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}}\lambda^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}} = \lambda^{\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q}} = \varepsilon_\mathfrak{q}.$$

Therefore, $\varepsilon_\mathfrak{q}$ is fixed by every $\tau \in \mathrm{Gal}(L(\lambda)/k)$, whence $\varepsilon_\mathfrak{q} \in K$, as claimed.

Let $\theta \in \mathbb{C}[\mathrm{Gal}(L/k)]$ be the "Stickelberger element" for the extension L/k, with respect to the exceptional set S. This element is characterized by the condition that

$$\chi(\theta) = L_S(\bar{\chi}, 0) \quad \text{for all } \chi \in \widehat{\mathrm{Gal}(L/k)}.$$

A classical result of Siegel [10] shows that the coefficients of θ are all rational. Moreover, the denominators are bounded, as shown by Barsky [1], Cassou-Noguès [2], and Deligne and Ribet [3], independently. Specifically, for any $A \in \mathbb{Z}[\mathrm{Gal}(L/k)]$ that annihilates the module $\mu(L)$ of roots of unity in L, they show that $A\theta$ has integral coefficients. In particular, taking $A = w_L$ gives the bound on the denominator.

Here we will consider elements of the form $A = \varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q}$, where $\mathfrak{q}$ is a place of k that is unramified in L, and does not divide w_L. It is easy to see that such an element indeed annihilates $\mu(L)$. Furthermore, we even have

Proposition 1.2. *Let L/k be an abelian extension of global fields, and S a finite set of places of k that contains all archimedean places, any places ramified in L, and any places that divide w_L. Then the annihilator of the* $\mathrm{Gal}(L/k)$*-module $\mu(L)$ is generated as a $\mathbb{Z}$-module by the elements $\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q}$, for all $\mathfrak{q} \notin S$.*

Proof. See [16, Chapitre IV, Lemme 1.1]. □

Accordingly, let $\mathfrak{q}$ be a place of k that is not in S and also does not divide $w_{L(\lambda)}$. Thus $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta$ has integral coefficients, so write it as $\sum_g n(g) \cdot g$. Since $\mathfrak{p}_0$ splits completely in the extension K/k, the decomposition group $G_{\mathfrak{P}_0}$ is contained in $H = \mathrm{Gal}(L/K)$. Let $r_{\mathfrak{P}_0}$ be the local reciprocity map at $\mathfrak{P}_0$, for the extension L/K. This is a map $r_{\mathfrak{P}_0} : K^*_{\mathfrak{P}_0} \to G_{\mathfrak{P}_0}$. Thus we can consider the composite map

$$K^* \hookrightarrow K^*_{\mathfrak{P}_0} \longrightarrow G_{\mathfrak{P}_0} \hookrightarrow \mathrm{Gal}(L/K),$$

which we will also denote by $r_{\mathfrak{P}_0}$.

We may now formulate the local Stark conjecture. See Gross' paper [7] for the original formulation, as well as Hayes [9] for a variant.

Conjecture 1.3. (The local Stark conjecture) *With the preceding notation, we have*

$$r_{\mathfrak{P}_0}(\varepsilon_{\mathfrak{q}}) = \prod_{h \in H} h^{n(h)}.$$

Conjecture 1.3 is an attempt to understand the Stark unit from a $\mathfrak{p}$-adic viewpoint. However, as remarked above, the Stark unit is not uniquely determined. The modified Stark unit is void of any such ambiguity, which explains its role in the conjecture.

The local Stark conjecture is known when the base field $k = \mathbb{Q}$, (see [6]) from the work of Gross and Koblitz [8]. It has also been proved by Hayes [9] in the function field case.

Connection to Hayes' formulation

The original formulation of the local Stark conjecture predicts the value of the local reciprocity map applied not only to the modified Stark unit, but also to all of its Galois conjugates as well. We briefly indicate here its connection with Conjecture 1.3, which is apparently weaker. In fact, we show that 1.3 implies Hayes' "slightly stronger" version [9, second form, (1.9)]. The equivalence to both of his versions, as well as Gross' original formulation is then clear.

We continue with the notation introduced above. Let $\widetilde{L} = L(\lambda)$, $\widetilde{G} = \mathrm{Gal}(\widetilde{L}/k)$, and let $A \in \mathbb{Z}[G]$ be an element that annihilates $\mu(L)$. From 1.2, A can be lifted to $\widetilde{A} \in \mathbb{Z}[\widetilde{G}]$, which annihilates $\mu(\widetilde{L})$. Hayes' stronger version of the conjecture is:

Conjecture 1.4. *If $\widetilde{A}$ is any such lift, then*

$$r_{\mathfrak{P}_0}(\lambda^{\widetilde{A}}) = \prod_{h\in H} h^{n(h)},$$

where $A\theta = \sum_{g\in G} n(g)\cdot g$.

As Hayes elucidates, this formulation also provides the values of the local reciprocity map on Galois conjugates of the modified Stark unit.

Proposition 1.5. *Conjecture 1.3 implies Conjecture 1.4.*

Proof. Let $\widetilde{A}$ be any lift of A to $\mathbb{Z}[\widetilde{G}]$ that annihilates $\mu(\widetilde{L})$. From 1.2, we may write $\widetilde{A} = \sum_{i=1}^{t} c_i(\varphi_{\mathfrak{q}_i} - \mathbf{N}\mathfrak{q}_i)$, where the $\mathfrak{q}_i$'s are places of k, but not in a finite set containing S. Then we find that $r_{\mathfrak{P}_0}(\lambda^{\widetilde{A}}) = \prod_{i=1}^{t} r_{\mathfrak{P}_0}(\varepsilon_{\mathfrak{q}_i})^{c_i}$. Now 1.3 evaluates each $r_{\mathfrak{P}_0}(\varepsilon_{\mathfrak{q}_i})$. Specifically, write $(\varphi_{\mathfrak{q}_i} - \mathbf{N}\mathfrak{q}_i)\theta = \sum_{g\in G} n_{\mathfrak{q}_i}(g)\cdot g$. Then 1.3 gives $r_{\mathfrak{P}_0}(\varepsilon_{\mathfrak{q}_i}) = \prod_{h\in H} h^{n_{\mathfrak{q}_i}(h)}$, so that $r_{\mathfrak{P}_0}(\lambda^{\widetilde{A}}) = \prod_{h\in H} h^{m(h)}$, where the exponent $m(h) = \sum_{i=1}^{t} c_i n_{\mathfrak{q}_i}(h)$. Finally, 1.4 follows by noting that the coefficients of $A\theta = \sum_{g\in G} m(g)\cdot g$ are also given by $m(g) = \sum_{i=1}^{t} c_i n_{\mathfrak{q}_i}(g)$. $\square$

This argument also shows that in Conjecture 1.3, a finite number of $\mathfrak{q}$'s may be omitted, without weakening the conjecture. This may be useful because, in practice, one often does not know the field $L(\lambda)$ explicitly, and thus one does not know $w_{L(\lambda)}$, but only an upper bound on it.

2. Statement of Results

Theorem 2.1. *Let L/k be a CM extension, i.e. k is totally real, L is totally complex and is quadratic over its maximal totally real subfield, K. Let S be any finite set of places of k that contains all the archimedean places, and all places ramified in L. Let ∞_k be one of the real places of k, and ∞_K a place K lying over it. Then the local Stark conjecture for $L/K/k$ at ∞_k is true.*

In this theorem, we can easily identify the Stark unit. In fact, in most cases, we can take $\varepsilon = 1$. Thus, Stark's original conjecture is not interesting in this situation. On the other hand, the truth of the local conjecture relies on some delicate parity information involving partial zeta functions. This is where the Deligne-Ribet congruences are used.

For any abelian extension L/k of number fields with Galois group G, we can consider the subgroup $G_1 \subseteq G$ generated by all "complex conjugations". Since G is abelian, G_1 is simply an elementary abelian 2-group of rank $\leq r = [k:\mathbb{Q}]$. For a general "large" abelian extension, where k is totally real of absolute degree r, one expects G_1 to have full rank. In some cases, G_1 will collapse; a CM extension is an extreme example, where G_1 has rank 1. We next consider the situation in which G_1 is "partially collapsed", i.e. has rank strictly less than r. Here we generally cannot determine the Stark unit. Nevertheless, we have the following consequence of the local Stark conjecture.

Theorem 2.2. *Let k be a totally real field of absolute degree $r = [k : \mathbb{Q}] \geq 2$, L/k an abelian extension with Galois group G, and suppose that $\#G_1 \leq 2^{r-1}$. Let S be a finite set of places of k that contains the archimedean places and any places ramified in L, and suppose also that $\#S \geq 3$. Let ∞_k be one of the real places of k that ramifies in L, K its decomposition field in the extension L/k, and ∞_K a place of K lying over it. Finally, let $\varepsilon \in K^*$ be the hypothetical Stark unit for K/k, chosen to be positive at ∞_K. If the local Stark conjecture for $L/K/k$ holds, then ε is a square in K.*

The theorem above generalizes a result of Dummit and Hayes to a much wider class of situations. The significance of the result, as they already note, is that the abelian part of Stark's conjecture holds automatically. It also simplifies the task of "recognizing" real numbers as elements of number fields, and simplifies some other computations, as we'll see in section 5.

To describe the relation to the Dummit-Hayes situation, it is helpful to separate the following ingredient that is implicit in their work.

Theorem 2.3. (Dummit-Hayes [4]) *With the notation of Theorem 2.2, suppose that, for almost all primes $\mathfrak{q}$ of k, the coefficients of $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta$ are all even. Then the local conjecture implies that the Stark unit ε is a square in K.*

In their situation, k is a totally real field of odd absolute degree $r = [k : \mathbb{Q}] > 1$, S is the set of archimedean places, and L is the "narrow" Hilbert class field. In this case, they find that $\theta = 0$, so the hypothesis in the above theorem holds trivially. In our more general situation, we will use the Deligne-Ribet congruences to obtain the necessary parity information.

In Theorem 2.2 above, we make an extra hypothesis that $\#S \geq 3$. This excludes two cases, namely $k = \mathbb{Q}$, $S = \{\infty, p\}$, and k a real quadratic field, with S being its two archimedean places. In these two cases, the statement of Theorem 2.2 generally fails; however, in both cases, Stark's conjecture and the local conjecture both hold, so this is better than our hypothetical theorem.

3. The 2-adic congruences of Deligne-Ribet

In this section, we review the 2-adic congruences of Deligne-Ribet. Their congruences are much more far-reaching than what we give here; however, this will suffice for our purposes.

Let k be a totally real number field of absolute degree $r = [k : \mathbb{Q}]$, and let $\mathfrak{f}$ be an ideal of k. Let $\mathfrak{f}_\infty$ be the (formal) product of all the infinite places of k, and let $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ be the ray class group modulo $\mathfrak{f}\mathfrak{f}_\infty$. For any function $\mathbf{e} : \mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty) \to \mathbb{C}$, the function

$$L(\mathbf{e}, s) = \sum_{\mathfrak{a}} \mathbf{e}([\mathfrak{a}]) N\mathfrak{a}^{-s}$$

converges for $\Re s > 1$ and extends to a meromorphic function on all of $\mathbb{C}$. If $\mathbf{e}$ takes its values in $\mathbb{Q}$, then the previously mentioned result of Siegel shows that $L(\mathbf{e}, n)$ is rational for every integer $n \leq 0$. For a prime $\mathfrak{q}$ not dividing $\mathfrak{f}$, let $\mathbf{e}_\mathfrak{q}$ denote the function on $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ defined by $\mathbf{e}_\mathfrak{q}(x) = \mathbf{e}([\mathfrak{q}]x)$. By class field theory, we may identify $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ with the Galois group $\mathrm{Gal}(k(\mathfrak{f}\mathfrak{f}_\infty)/k)$, where $k(\mathfrak{f}\mathfrak{f}_\infty)$ is the ray class field modulo $\mathfrak{f}\mathfrak{f}_\infty$. Via this identification, we may consider complex conjugations of $k(\mathfrak{f}\mathfrak{f}_\infty)$ as elements of $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$.

3.1. Theorem. (Deligne-Ribet [3]) *Suppose that* $\mathbf{e}$ *is an odd function, i.e.* $\mathbf{e}(cx) = -\mathbf{e}(x)$, *for every complex conjugation* c, *and suppose that* $\mathbf{e}$ *takes values in* $\mathbb{Z}$. *Then for any prime* $\mathfrak{q}$ *not dividing* $2\mathfrak{f}$,

$$\Delta_\mathfrak{q}(\mathbf{e}) = L(\mathbf{e}, 0) - N\mathfrak{q}L(\mathbf{e}_\mathfrak{q}, 0)$$

is an integer divisible by 2^{r-1}. *Moreover, it is divisible by* 2^r, *except in the* exceptional case, *in which all of the following conditions are satisfied:*

- *the finite part of the conductor,* $\mathfrak{f}$, *is trivial,*
- *all units of* k *have absolute norm* $+1$ *down to* $\mathbb{Q}$,
- *the extension* k'/k *obtained by taking square roots of all totally positive units of* k *is a quadratic extension,*
- *the prime* $\mathfrak{q}$ *is inert in* k', *and*
- *the sum* $\delta(\mathbf{e}) = \sum_{x \in \mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)/C} \mathbf{e}(x)$ *is odd, where* $C \subseteq \mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ *is the subgroup generated by all complex conjugations. Note that for* $x \in \mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)/C$, *the parity of* $\mathbf{e}(x)$ *is well-defined.*

In the exceptional case, $\Delta_\mathfrak{q}(\mathbf{e})$ *is not divisible by* 2^r.

4. Proofs

First we concern ourselves with the order of vanishing of L-functions at $s = 0$. This is handled by the following.

Proposition 4.1. *For a character* χ *of* $\mathrm{Gal}(K/k)$, *the order of vanishing of* $L_S(\chi, s)$ *at* $s = 0$ *is given by*

$$\mathrm{ord}_{s=0} L_S(\chi, s) = \begin{cases} \#S - 1, \ \text{if } \chi = \chi_0, \text{ and} \\ \\ \text{the number of places in } S \\ \text{that split completely in } K^{\ker \chi}, \text{ otherwise.} \end{cases}$$

Proof. See [16, Chapitre I, Proposition 3.4] for a more general formula which gives the order of vanishing of Artin L-functions. □

We now show that Stark's conjecture holds (relatively easily) for K/k in the situation of Theorem 2.1.

Proposition 4.2 *Let* k *be a totally real field of absolute degree* $r = [k : \mathbb{Q}] > 1$, *and* K/k *be an abelian extension, with* K *also totally real. Then Stark's conjecture (1.1) holds for* K/k *with respect to any appropriate set* S. *Moreover, the Stark unit can be taken as* $\varepsilon = 1$, *unless* $r = 2$, *and* S *contains only the two archimedean places. Furthermore, in this* *case, we may take $\varepsilon = u^{h_k/d}$ as the Stark unit, where u is a totally positive fundamental unit of k, and $d = [K : k]$.*

Proof. Each of the r archimedean places splits completely in K. Therefore, if χ is a non-trivial character, $L_S(\chi, s)$ vanishes to order at least r, so that $L'_S(\chi, s) = 0$. For the trivial character, $L_S(\chi, s)$ vanishes to order $\#S - 1$. Therefore, if $\#S \geq 3$, then we have $L'_S(\chi, s) = 0$ for all characters χ, so we may take $\varepsilon = 1$ as the Stark unit.

If $\#S = 2$, this forces $r = [k : \mathbb{Q}] = 2$, and S must consist only of the two archimedean places of k. In this case, we have $L'(\chi, s) = 0$ for non-trivial characters, while $L'(\chi_0, s) = -h_k R_k / w_k$, from formula (1) of the introduction. Thus we may take $\varepsilon = u^{h_k/d}$, where u is a fundamental unit of k satisfying $|u|_{\mathfrak{P}_0} > 1$. Note that, in this case, K/k is everywhere unramified, so that $d = [K : k]$ divides h_k.

The abelian condition in (1.1) is also satified. If $\varepsilon = 1$, this is clear. Otherwise, note that $w_K = 2$, since K is totally real. Then $\varepsilon = u^{h_k/d}$, so that $K(\sqrt{\varepsilon}) \subseteq K(\sqrt{u})$. which is the compositum of K and $k(\sqrt{u})$, both of which are abelian extensions of k. $\square$

Proposition 4.3. *Let k be totally real of absolute degree $r > 1$, L/k an abelian extension with Galois group G, and suppose that G_1, the subgroup generated by complex conjugations, has order $\leq 2^{r-1}$. Let S be any finite set of places of k, containing all the archimedean places, as well as any ramified in L, and let $\theta \in \mathbb{Q}[G]$ be the Stickelberger element, relative to the set S. If $\mathfrak{q}$ is any prime of k, not in S, nor dividing w_L, then all coefficients of $(\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q})\theta$ are even, unless*

(1) *$r = 2$,*
(2) *S contains only the 2 archimedean places of k,*
(3) *k has a fundamental unit, u, that is totally positive,*
(4) *h_k/d is odd, where $d = [L^{G_1} : k]$ is the degree over k of the fixed field of G_1 (note that L^{G_1} is everywhere unramified, so that d divides h_k), and*
(5) *$\mathfrak{q}$ is inert in the extension $k' = k(\sqrt{u})$.*

When all 5 conditions are satisfied, then the coefficients of $(\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q})\theta$ are odd.

Proof. Let G_2 be the subgroup of G generated by products of pairs of complex conjugations, $\sigma_1\sigma_2$. Clearly, $(G_1 : G_2) = 1$ or 2. If this index is 1, then there is a relation $\sigma_1\sigma_2\cdots\sigma_t = 1$, where t is odd. In that case, if χ is any character of G, then $\chi(\sigma) = 1$ for some complex conjugation σ. Thus $L(\chi, 0) = 0$ for all χ, whence $\theta = 0$. Therefore, all coefficients of $(\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q})\theta$ are even, as claimed. Also, conditions (1) – (5) cannot all hold in this case.

So now suppose that $(G_1 : G_2) = 2$. By Fourier inversion, the Stickelberger element is $\theta = \sum_{g \in G} m(g) \cdot g$, where the coefficient $m(g) = \frac{1}{\#G} \sum_{\chi \in \widehat{G}} \chi(g) L(\chi, 0)$. Let $g_0 \in G$ be an arbitrary element. The coefficient of g_0 in $(\varphi_\mathfrak{q} - \mathbf{N}\mathfrak{q})\theta$ is

$$n(g_0) = m(g_0\varphi_\mathfrak{q}^{-1}) - \mathbf{N}\mathfrak{q} m(g_0) = \frac{1}{\#G} \sum_{\chi \in \widehat{G}} \left(\chi(\varphi_\mathfrak{q}^{-1}) - \mathbf{N}\mathfrak{q}\right) \chi(g_0) L(\chi, 0).$$

Since $L(\chi, 0) = 0$ unless χ is odd, i.e. $\chi(\sigma) = -1$ for every complex conjugation σ, we need only consider odd χ in the above sum. Therefore,

$$n(g_0) = \frac{1}{\#G} \sum_{\chi \,\mathrm{odd}} \left(\chi(\varphi_{\mathfrak{q}}^{-1}) - \mathbf{N}\mathfrak{q}\right) \chi(g_0) L(\chi, 0).$$

Let $\mathfrak{f}$ be an ideal of k that is divisible by the finite part of the conductor of L/k, as well as every finite prime in S, but not by any place not in S. Let $\mathfrak{f}_\infty$ be the formal product of the archimedean places of k, and $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ the ray class group modulo $\mathfrak{f}\mathfrak{f}_\infty$. Class field theory provides an isomorphism $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty) \to \mathrm{Gal}(k(\mathfrak{f}\mathfrak{f}_\infty)/k)$, and we have a natural projection $\mathrm{Gal}(k(\mathfrak{f}\mathfrak{f}_\infty)/k) \to G$. Define a function $\mathbf{e} : G \to \mathbb{Z}$ by

$$\mathbf{e}(x) = \begin{cases} 1 & \text{if } x \in g_0^{-1} G_2 \\ -1 & \text{if } x \in \sigma g_0^{-1} G_2 \\ 0 & \text{otherwise,} \end{cases}$$

where σ is an arbitrary complex conjugation, so that σG_2 is the non-identity coset of G_2 in G_1. We may also consider $\mathbf{e}$ as a function on $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty)$ by composing with the map $\mathcal{S}(\mathfrak{f}\mathfrak{f}_\infty) \to G$. By construction, $\mathbf{e}$ is an odd function. Furthermore, we claim that

$$\mathbf{e} = \frac{1}{(G : G_1)} \sum_{\chi \,\mathrm{odd}} \chi(g_0) \chi. \tag{2}$$

To see why, note that every odd character is trivial on G_2, and thus is inflated from a character on G/G_2. Of those characters inflated from G/G_2, exactly half are non-trivial on the non-identity coset of G_2 in G_1. These are precisely the odd characters, so there are $\frac{1}{2}(G : G_2) = (G : G_1)$ of them. It is now clear that (2) holds, when applied to any element in $g_0^{-1} G_2$ or in $\sigma g_0^{-1} G_2$. So suppose that g is some element not in $g_0^{-1} G_1$. Then there is a character ψ, trivial on G_1, but not on the element $g_0 g$. Multiplication by ψ permutes the odd characters, so we have

$$\sum_{\chi \,\mathrm{odd}} \chi(g_0)\chi(g) = \sum_{\chi \,\mathrm{odd}} \psi\chi(g_0)\psi\chi(g) = \psi(g_0 g) \sum_{\chi \,\mathrm{odd}} \chi(g_0)\chi(g).$$

Since $\psi(g_0 g) \neq 1$, the sum is 0, and therefore equation (2) holds on all of G. In a similar way, we have

$$\mathbf{e}_{\mathfrak{q}} = \frac{1}{(G : G_1)} \sum_{\chi \,\mathrm{odd}} \chi(\varphi_{\mathfrak{q}}^{-1} g_0) \chi. \tag{3}$$

By the Deligne-Ribet theorem, $\Delta_{\mathfrak{q}}(\mathbf{e}) = L(\mathbf{e}_{\mathfrak{q}}, 0) - \mathbf{N}\mathfrak{q} L(\mathbf{e}, 0)$ is divisible by 2^{r-1}, and is divisible by 2^r if we're not in the exceptional case. From equations (2) and (3), we calculate

$$\begin{aligned}
\Delta_{\mathfrak{q}}(\mathbf{e}) &= L(\mathbf{e}_{\mathfrak{q}},0) - \mathbf{N}\mathfrak{q}L(\mathbf{e},0) \\
&= \frac{1}{(G:G_1)}\sum_{\chi\,\mathrm{odd}} \chi(\varphi_{\mathfrak{q}}^{-1}g_0)L_S(\chi,0) - \frac{\mathbf{N}\mathfrak{q}}{(G:G_1)}\sum_{\chi\,\mathrm{odd}}\chi(g_0)L_S(\chi,0) \\
&= \frac{\#G_1}{\#G}\sum_{\chi\,\mathrm{odd}}\left(\chi(\varphi_{\mathfrak{q}}^{-1}) - \mathbf{N}\mathfrak{q}\right)\chi(g_0)L_S(\chi,0) \\
&= \#G_1 n(g_0).
\end{aligned}$$

Thus, if we're not in the "exceptional case" of the Deligne-Ribet theorem, then $n(g_0)$ is divisible by $2^r/\#G_1$, and is therefore even, as claimed.

In the exceptional case, $\Delta_{\mathfrak{q}}(\mathbf{e})$ is exactly divisible by 2^{r-1}. We claim that $\#G_1 = 2$ in the exceptional case. Firstly, the finite part of the conductor of $\mathbf{e}$ is trivial, i.e. $\mathfrak{f} = 1$. In particular, this means that S contains only archimedean places. Secondly, the units of k have all possible signatures that have norm $+1$ down to $\mathbb{Q}$. We have an exact sequence

$$1 \longrightarrow U_k^+ \longrightarrow U_k \longrightarrow \{\pm 1\}^r \longrightarrow \mathcal{S}(\mathfrak{f}_\infty) \longrightarrow \mathcal{S}(1) \longrightarrow 0$$

where U_k is the unit group of k, U_k^+ is the subgroup of totally positive units, $U_k \to \{\pm 1\}^r$ is the "signature" map, and $\mathcal{S}(\mathfrak{m})$ is the ray class group modulo $\mathfrak{m}$. The condition on the signatures says that the cokernel of the signature map has order 2. Thus the kernel of $\mathcal{S}(\mathfrak{f}_\infty) \to \mathcal{S}(1)$ also has order 2. However, this kernel corresponds exactly to G_1, under the reciprocity map of class field theory. This proves the claim.

Therefore, $n(g_0)$ is even, unless $r = 2$ and we're in the "exceptional case" of the Deligne-Ribet theorem, in which case it is odd. In this situation, the finite part of the conductor, $\mathfrak{f}$, is trivial. Therefore, S contains only the 2 archimedean places of k. Secondly, all units of k have norm $+1$ down to $\mathbb{Q}$. This means that k has a fundamental unit, u, that is totally positive. Moreover, $\mathfrak{q}$ is inert in the extension $k' = k(\sqrt{u})$. Lastly, $\delta(\mathbf{e})$ is odd. To interpret this condition, we must consider $\mathbf{e}$ as a function on $\mathcal{S}(\mathfrak{f}_\infty)$. From the exact sequence above, we see that $\#\mathcal{S}(\mathfrak{f}_\infty) = 2h_k$, so that $\mathcal{S}(\mathfrak{f}_\infty) \to G$ is an h_k/d to 1 map. Now it follows that $\delta(\mathbf{e}) = h_k/d \bmod 2$. Therefore, we have shown that $n(g_0)$ is even, unless conditions (1) through (5) all hold, in which case it is odd. □

Now we are in a position to prove our main results.

Proof of Theorem 2.1. We place ourselves in the context of 2.1. So let k be totally real of absolute degree $r > 1$ and L/k a CM extension with maximal totally real subfield K. Let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) = \{1, \sigma\}$. Let S be a finite set of places of k, which includes all r archimedean places, and any others ramified in L. Also let ∞_k be one of the real places of k, ∞_K a place of K lying over it, and ∞_L the unique place of L over ∞_K. Finally, let $\mathfrak{q}$ be a place of k that is not in S, and also does not divide w_L.

Let ε be the Stark unit for K/k, with respect to the set S, which is known to exist, from 4.2, and $\varepsilon_{\mathfrak{q}}$ the modified Stark unit. Let θ be the Stickelberger element for the extension

L/k, with respect to S, and write $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta = \sum_g n(g)g$. Then the local Stark conjecture (1.3) predicts that

$$r_{\infty_K}(\varepsilon_{\mathfrak{q}}) = \prod_{h \in H} h^{n(h)}.$$

Since $H = \{1, \sigma\}$, the right hand side simplifies to $\sigma^{n(\sigma)}$, which is either 1 or σ, depending on the parity of the coefficient $n(\sigma)$. The left hand side, $r_{\infty_K}(\varepsilon_{\mathfrak{q}})$ is either 1 or σ, depending on the sign of $\varepsilon_{\mathfrak{q}}$, with respect to the real embedding corresponding to ∞_K. Thus we must show that $\mathrm{sgn}_{\infty_K}(\varepsilon_{\mathfrak{q}}) = +1$ if and only if the coefficient $n(\sigma)$ is even.

First suppose that $\#S > 2$. Then 4.2 shows that $\varepsilon = 1$, whence $\varepsilon_{\mathfrak{q}} = 1$. Also, 4.3 shows that $n(\sigma)$ is even, so we are done in this case.

Now suppose that $\#S = 2$, which requires k to be real quadratic, and S to consist only of the two archimedean places. Moreover, k has a totally positive fundamental unit, u. This is only determined up to inversion, so we choose u by requiring it to be greater than 1 with respect to the real embedding corresponding to ∞_k. Then we may take $\varepsilon = u^{h_k/d}$, where $d = [K : k]$, so that

$$\varepsilon_{\mathfrak{q}} = \left[\left(\sqrt{u}\right)^{\varphi_{\mathfrak{q}}-1} u^{(1-\mathbf{N}\mathfrak{q})/2}\right]^{h_k/d}.$$

Now $(\sqrt{u})^{\varphi_{\mathfrak{q}}-1} = \pm 1$, depending upon whether $\mathfrak{q}$ splits or is inert in the extension $k' = k(\sqrt{u})$. Also, $\mathrm{sgn}_{\infty_K}(u) = +1$, so the power of u can be ignored. Therefore, $\mathrm{sgn}_{\infty_K}(\varepsilon_{\mathfrak{q}}) = +1$, unless $\mathfrak{q}$ is inert in k' and h_k/d is odd. However, these are precisely conditions (4) and (5) of 4.3 (conditions (1), (2) and (3) are already satisfied), so $n(\sigma)$ is even precisely when $\mathrm{sgn}_{\infty_K}(\varepsilon_{\mathfrak{q}}) = +1$. Thus we have proven the refined Stark conjecture for CM extensions. □

Proof of Theorem 2.2. We place ourselves in the context of 2.2. Let k be totally real of absolute degree $r \geq 2$, and let L/k be an abelian extension with Galois group G. Let S be a finite set of places of k that contains all the archimedean places, and any ramified in L. We also assume that $\#S \geq 3$. Let ∞_k be an archimedean place of k that ramifies in L; let σ be the corresponding complex conjugation, K its fixed field, and ∞_K one of the places of K over ∞_k. Let $\varepsilon \in K^*$ be the hypothetical Stark unit for the extension K/k, with respect to the exceptional set S. We choose ε so that it is positive at ∞_K.

Suppose that G_1, the subgroup of G generated by all complex conjugations does not have full rank, i.e. has order less than 2^r. We must show that ε is a square in K. Let $\mathfrak{Q}$ be any place of K having degree 1 over k, and which also does not lie over any place in S, nor divide w_L. Note that the set of such $\mathfrak{Q}$ has density 1. Let $\mathfrak{q}$ be the place of k under $\mathfrak{Q}$. We are assuming that the local Stark conjecture holds, so that $r_{\infty_K}(\varepsilon_{\mathfrak{q}}) = \prod_{h \in H} h^{n(h)}$, where $H = \mathrm{Gal}(L/K) = \{1, \sigma\}$, and $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta = \sum_{g \in G} n(g)g$. Since H has order 2, the product simplifies to $\sigma^{n(\sigma)}$. Moreover, the coefficient $n(\sigma)$ is even, from 4.3. Thus $\varepsilon_{\mathfrak{q}}$ is positive at ∞_K. However,

$$\varepsilon_{\mathfrak{q}} = \left(\sqrt{\varepsilon}\right)^{\varphi_{\mathfrak{q}}-1} \varepsilon^{(1-\mathbf{N}\mathfrak{q})/2}.$$

The factor $(\sqrt{\varepsilon})^{\varphi_{\mathfrak{q}}-1}$ is ± 1, depending on whether $\mathfrak{Q}$ splits or is inert in the extension $K' = K(\sqrt{\varepsilon})$. Also, ε was chosen to be positive at ∞_K, so the second factor does not affect the sign. Therefore, $\mathfrak{Q}$ splits in K', and since the set of such $\mathfrak{Q}$ has density 1, this means that the extension K'/K has degree 1, i.e. ε is a square in K. The proof of Theorem 2.2 is complete. □

Here we have used the same argument as Dummit-Hayes [4]. In their situation, $\theta = 0$, in fact, they are in the situation that $(G_1 : G_2) = 1$, in the notation of (4.3). Thus, the evenness of the coefficients of $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta$ is trivial in their situation, whereas, in our more general setting, we needed the 2-adic congruences of Deligne-Ribet.

5. A Computational Example

Let k be the totally real cubic field $\mathbb{Q}(\alpha)$ where $\alpha^3-\alpha^2-4\alpha+3 = 0$. It has discriminant 257 (and thus is not normal over $\mathbb{Q}$), its ring of integers is $\mathbb{Z}[\alpha]$, it has class number 1, and its unit group is generated by $\alpha - 1, \alpha - 2$ and -1. The three real places ∞_1, ∞_2 and ∞_3 correspond to the real embeddings

$$\alpha \mapsto 2.19869124351\ldots \qquad \alpha \mapsto 0.71353793496\ldots \qquad \alpha \mapsto -1.91222917848\ldots$$

respectively.

The rational prime 907 splits completely in k as $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''$. We only concern ourselves here with $\mathfrak{p} = (-2\alpha^2 + \alpha + 16)$. The ray class group modulo $\mathfrak{p}\infty_1\infty_2\infty_3$ is $(\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$; let L be the corresponding ray class field and $G = \mathrm{Gal}(L/k) \cong \mathcal{S}(\mathfrak{p}\infty_1\infty_2\infty_3)$. The ray class group modulo $\mathfrak{p}\infty_2\infty_3$ is cyclic of order 6; let K be the corresponding ray class field. Therefore, ∞_1 ramifies in L, and K is the fixed field of the corresponding complex conjugation. Let G_1 be the subgroup of G generated by all complex conjugations. Since $\#G$ is not divisible by 8, G_1 does not have full rank; in fact, $\mathcal{S}(\mathfrak{p})$ has order 3, so G_1 has order 4.

Let ∞_k be the archimedean place ∞_1, and let ∞_K be one of the places of K lying over ∞_k. Let $\varepsilon \in K^*$ be the hypothetical Stark unit for K/k, chosen to be positive at ∞_K. We are in the situation of Theorem 2.2, so ε should be a square in K.

The quadratic subfield of K/k can be determined more or less by inspection. In fact, the ray class group $\mathcal{S}(\infty_2\infty_3)$ has order 2, so the corresponding ray class field is this intermediate quadratic field. Then we find that it is $k(\sqrt{\alpha-1})$: this extension is unramified at 2 because $(\alpha^2+\alpha+\sqrt{\alpha-1})/2$ is an algebraic integer. We also note that $\mathfrak{p}$ splits in this extension.

The ray class group $\mathcal{S}(\mathfrak{p}\infty_2\infty_3)$ is generated by $\mathfrak{c}$, the class of (α^2-3). Let $\sigma \in \mathrm{Gal}(K/k)$ be the corresponding generator of the Galois group, and let χ be a generator of $\widehat{\mathrm{Gal}(K/k)}$ such that $\chi(\sigma) = e^{\pi i/3}$. Using Pari/GP functionality, we computed to 70 decimal places the values

$$L'_S(\chi, 0) = -13.991520685060219482871 1\ldots - i16.434068079480184753847 3\ldots$$

$$L'_S(\chi^5, 0) = -13.991520685060219482871 1\ldots + i16.434068079480184753847 3\ldots$$

where $S = \{\mathfrak{p}, \infty_1, \infty_2, \infty_3\}$. The corresponding values for χ_0, χ^2, χ^3 and χ^4 are 0, from 4.1 above; for χ^3, we use the observation that $\mathfrak{p}$ splits in $k(\sqrt{\alpha-1})$. Then, by Fourier inversion, we compute $|\varepsilon^{\sigma^j}|_{\infty_K} = \exp(-(\zeta^{-j}L'_S(\chi,0)+\zeta^j L'_S(\chi^5,0))/3)$, where $\zeta = e^{\pi i/3}$. This gives

$$|\varepsilon|_{\infty_K} = 11245.017905578404131419698010914229396869749699733679537102 1\ldots,$$

$$|\varepsilon^{\sigma}|_{\infty_K} = 1400099.4922060985114287168762610205589025535118451986628069\ldots,$$

$$|\varepsilon^{\sigma^2}|_{\infty_K} = 124.50842710633125690819845447034081302561526732561117276 29\ldots,$$

$$|\varepsilon^{\sigma^3}|_{\infty_K} = 0.00008892827102604453528778309346194843570533045398947606 7\ldots,$$

$$|\varepsilon^{\sigma^4}|_{\infty_K} = 0.00000071423495656321346529134191553159828523553258654853 4\ldots,$$

$$|\varepsilon^{\sigma^5}|_{\infty_K} = 0.0080315848753433491312187671031841493173678493805427089 58\ldots.$$

We have chosen ε to be positive with respect to the embedding defined by ∞_K. In fact, we claim that ε is totally positive. To see this, let $\tau \in \mathrm{Gal}(K/k)$ be an arbitrary element, and write $\varepsilon^{1-\tau} = (\lambda^{1-\widetilde{\tau}})^2$, where $\lambda = \sqrt{\varepsilon}$ and $\widetilde{\tau}$ is any lift of τ to $\mathrm{Gal}(L/k)$. One shows that $\lambda^{1-\widetilde{\tau}} \in K$, by a calculation similar to that which shows that the modified Stark unit is in K. Thus, for any τ, we have $\varepsilon^{1-\tau} \in (K^*)^2$, which shows that ε is totally positive.

Let s_m be the m-th symmetric function of the values above. Using Pari/GP, we were able to identify the numeric values as elements of k:

$$s_1 = s_5 = 231185\alpha^2 + 277121\alpha - 315439,$$

$$s_2 = s_4 = 2607530650\alpha^2 + 3125624158\alpha - 3557840132,$$

$$s_3 = 321096877176\alpha^2 + 384896014993\alpha - 438120010876,$$

$$s_6 = 1.$$

Thus ε is a root of the symmetric sextic polynomial

$$\begin{aligned} f(X) = X^6 &-(231185\alpha^2 + 277121\alpha - 315439)X^5 \\ &+(2607530650\alpha^2 + 3125624158\alpha - 3557840132)X^4 \\ &-(321096877176\alpha^2 + 384896014993\alpha - 438120010876)X^3 \\ &+(2607530650\alpha^2 + 3125624158\alpha - 3557840132)X^2 \\ &-(231185\alpha^2 + 277121\alpha - 315439)X + 1. \end{aligned}$$

Over k, $f(X)$ is irreducible, but $f(X^2)$ factors as $g(X)g(-X)$, where

$$\begin{aligned} g(X) = X^6 &-(213\alpha^2 + 256\alpha - 292)X^5 \\ &+(22930\alpha^2 + 27484\alpha - 31286)X^4 \\ &-(231611\alpha^2 + 277633\alpha - 316021)X^3 \\ &+(22930\alpha^2 + 27484\alpha - 31286)X^2 \\ &-(213\alpha^2 + 256\alpha - 292)X + 1. \end{aligned}$$

(If we had worked with $\sqrt{\varepsilon}$ instead, 40 decimal places of accuracy would be sufficient to identify the symmetric functions of its Galois conjugates, and we would have found the above polynomial directly.)

Now $\eta = \sqrt{\varepsilon} + 1/\sqrt{\varepsilon}$ is a root of the cubic polynomial

$$h(X) = X^3 - (213\alpha^2 + 256\alpha - 292)X^2 + (22930\alpha^2 + 27484\alpha - 31289)X \\ - (231185\alpha^2 + 277121\alpha - 315437).$$

A root of this equation should define the ray class field modulo $\mathfrak{p}$ over k. It behooves us to find a simpler defining polynomial. Take the product of $h(X)$ with its $\mathbb{Q}$-conjugates to get a degree 9 polynomial for η over $\mathbb{Q}$:

$$X^9 - 1297X^8 + 135361X^7 - 913746X^6 - 4958790X^5 - 2179354X^4 \\ + 14965024X^3 + 22396167X^2 + 9433638X + 386839.$$

In Pari/GP, we compute a simpler defining polynomial for the field $\mathbb{Q}(\eta)$, which is

$$X^9 - 3X^8 - 8X^7 + 32X^6 - 14X^5 - 32X^4 + 24X^3 + 4X^2 - 6X + 1.$$

It is worth emphasizing that the corresponding calculation for the minimal polynomial of $\varepsilon + 1/\varepsilon$ is dramatically more complex, because the height of the polynomial is much larger. Over k, the 9-th degree polynomial above factors as a product of an irreducible cubic and sextic, the cubic being

$$X^3 - (\alpha^2 - 2)X^2 + (\alpha^2 - 5)X + 1.$$

Let β be a root of this polynomial. The discriminant of the polynomial is $(-2\alpha^2 + \alpha + 16)^2$, which shows that the extension $k(\beta)/k$ is Galois, cyclic of order 3. Moreover, it is unramified outside of $\mathfrak{p}$: it is unramified at other finite places because they don't divide the discriminant, and unramified at archimedean places because it is a normal extension of odd degree. This proves that $k(\beta)$ is the ray class field modulo $\mathfrak{p}$, and the composite field $k(\beta, \sqrt{\alpha - 1})$ is K, the ray class field modulo $\mathfrak{p}\infty_2\infty_3$. Finally, the Stark unit is

$$\varepsilon = \left(\frac{a_2\beta^2 + a_1\beta + a_0 + \left(b_2\beta^2 + b_1\beta + b_0\right)\sqrt{\alpha - 1}}{2} \right)^2$$

where the coefficients are

$$a_2 = 31\alpha^2 + 37\alpha - 42,$$

$$a_1 = -15\alpha^2 - 18\alpha + 20,$$

$$a_0 = -\alpha^2 - \alpha,$$

$$b_2 = 22\alpha^2 + 26\alpha - 31,$$

$$b_1 = 11\alpha^2 + 12\alpha - 17,$$

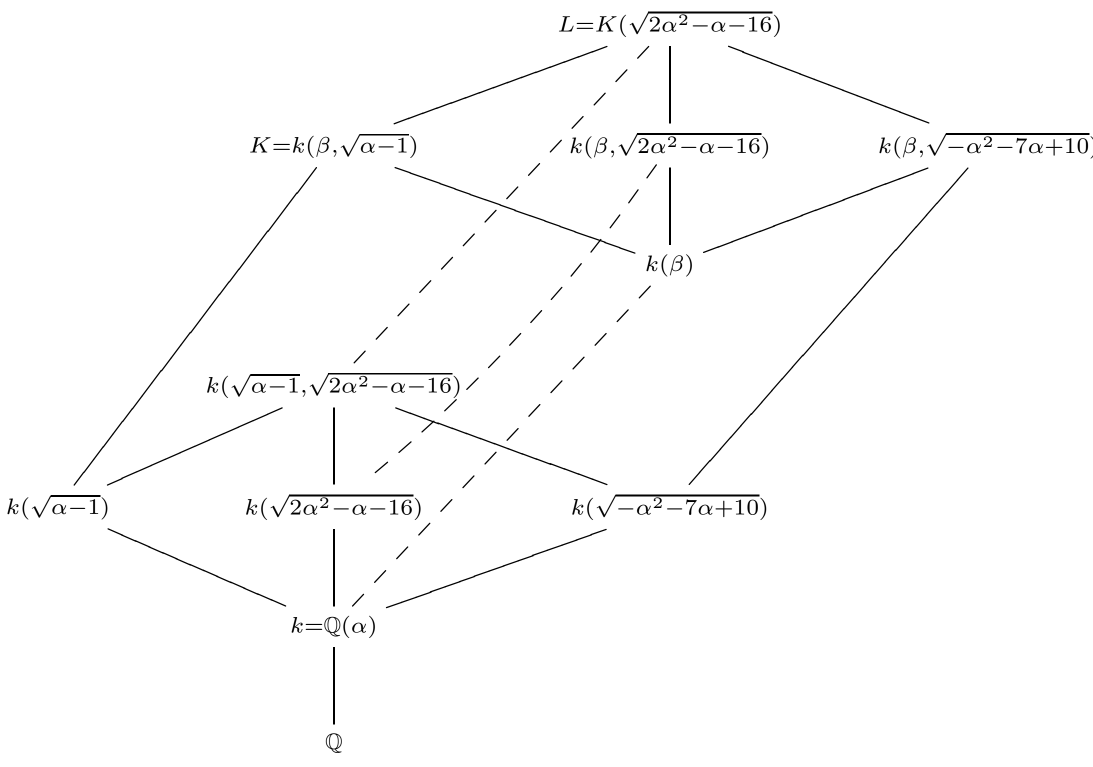
$$b_0 = -18\alpha^2 - 20\alpha + 27.$$

The action of the Galois group is given by

$$\sigma(\sqrt{\alpha - 1}) = -\sqrt{\alpha - 1} \qquad \text{and} \qquad \sigma(\beta) = \beta^2 - (\alpha^2 - 2)\beta + (\alpha^2 - 4).$$

The numerical values of ε and its conjugates, under this action, all agree with the values computed above, to within 10^{-66}.

To complete the entire picture of fields involved, it remains to identify the field L. There are three quadratic subfields of L/k; one is $k(\sqrt{\alpha - 1})$, which is contained in K. For either of the other two, its composite with K is L. They are unramified outside of S, so it is straightforward to find them. We find that $k(\sqrt{2\alpha^2 - \alpha - 16})$ is unramified outside of S: it is unramified at 2, since $(\alpha^2 + \alpha + 1 + \sqrt{2\alpha^2 - \alpha - 16})/2$ is an algebraic integer. Therefore, this gives one of the quadratic fields, and the other is $k(\sqrt{-\alpha^2 - 7\alpha + 10})$, since $(\alpha - 1)(2\alpha^2 - \alpha - 16) = -\alpha^2 - 7\alpha + 10$. Thus we have the following diagram of fields.



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