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# GROSS' CONJECTURE FOR EXTENSIONS RAMIFIED OVER THREE POINTS OF $\mathbb{P}^1$

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Abstract. B. Gross has formulated a conjectural generalization of the class number formula. Suppose $L/K$ is an abelian extension of global fields with Galois group $G$. A generalized Stickelberger element $\theta \in \mathbb{Z}[G]$ is constructed from special values of $L$-functions at $s = 0$. Gross' conjecture then predicts some $I$-adic information about $\theta$, where $I \subseteq \mathbb{Z}[G]$ is the augmentation ideal. In this paper, we prove (under a mild hypothesis) the conjecture for the maximal abelian extension of the rational function field $\mathbb{F}_q(X)$ that is unramified outside a set of three degree 1 places.

## 1. Introduction

Let $K$ be a global field, i.e. either a finite extension of $\mathbb{Q}$, or a finite, separable extension of $\mathbb{F}_q(X)$. Let $S$ be any finite, non-empty set of places of $K$. We also require, in the number field case, that $S$ contains all the archimedean places. We consider the $S$-zeta function

$$\zeta_S(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_S} (\mathbf{N}\mathfrak{a})^{-s},$$

where the summation ranges over all ideals $\mathfrak{a}$ in the ring $\mathcal{O}_S$ of $S$-integers. This series definition makes sense only for $\Re(s) > 1$, where the series converges absolutely. However, it is well-known that this function has an analytic continuation to the whole complex plane, with only a simple pole at $s = 1$.

In the Taylor series expansion for $\zeta_S(s)$, at $s = 0$, the leading term has a very simple form. Specifically, we have

$$\zeta_S(s) = -\frac{h_S R_S}{w_S} s^n + O(s^{n+1}) \quad \text{near } s = 0, \tag{1.1}$$

where $h_S$ is the class number of $\mathcal{O}_S$, $R_S$ is the $S$-regulator, $w_S$ is the number of roots of unity in $\mathcal{O}_S$ (which does not actually depend upon $S$) and $n = \#S - 1$, which is the rank of the unit group, $\mathcal{O}_S^*$, by Dirichlet's unit theorem.

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In order to state Gross' conjecture, we need a formula such as (1.1) above, but which has no denominator. Towards this end, we introduce an auxiliary set, $T$, of places of $K$. For now, we only insist that $T$ is finite and disjoint from $S$, but we will make further restrictions below. We introduce the modified zeta function

$$\zeta_{S,T}(s) = \zeta_S(s) \prod_{\mathfrak{q} \in T} \left(1 - \mathbf{N}\mathfrak{q}^{1-s}\right).$$

The auxiliary factors are all regular and non-zero at $s = 0$. We will see that the leading term of the Taylor series of $\zeta_{S,T}(s)$ also has a simple form, namely

$$\zeta_{S,T}(s) = (-1)^{\#T-1} \frac{h_{S,T} R_{S,T}}{w_{S,T}} s^n + O(s^{n+1}) \quad \text{near } s = 0. \tag{1.2}$$

Each term in this formula has an interpretation analogous to the corresponding term in formula (1.1). Here $h_{S,T}$ is the order of the ray class group modulo $T$ (i.e. modulo the conductor which is the product of primes in $T$), $R_{S,T}$ is the regulator of $U_{S,T}$, the subgroup of units that are $\equiv 1 \bmod T$ (i.e. are congruent to 1 modulo every place in $T$), and $w_{S,T}$ is the number of roots of unity in $U_{S,T}$, or equivalently, the order of its torsion subgroup.

To prove formula (1.2) above, note that we have an exact sequence

$$1 \longrightarrow U_{S,T} \longrightarrow U_S \longrightarrow \prod_{\mathfrak{q} \in T} \kappa(\mathfrak{q})^* \longrightarrow \mathcal{C}l_{S,T} \longrightarrow \mathcal{C}l_S \longrightarrow 1$$

where $\kappa(\mathfrak{q})$ is the residue field at $\mathfrak{q}$, $\mathcal{C}l_S$ is the class group of $\mathcal{O}_S$, and $\mathcal{C}l_{S,T}$ is the ray class group modulo $T$. This shows that $(U_S : U_{S,T}) \frac{h_{S,T}}{h_S} = \prod_{\mathfrak{q} \in T} (\mathbf{N}\mathfrak{q} - 1)$. Moreover, we have $(U_S : U_{S,T}) = \frac{R_{S,T}}{R_S} \cdot \frac{w_S}{w_{S,T}}$. These last two equations, along with (1.1) and the definition of $\zeta_{S,T}$ give (1.2).

Thus, to achieve a formula without a denominator, we need only insist that $w_{S,T} = 1$, which imposes only mild conditions on $T$. Henceforth we make this assumption on $T$.

## Gross' Conjecture

Let $L$ be a finite abelian extension of $K$ with Galois group $G$. Consider $S$ as in the previous section, but now we will impose another condition, namely that $S$ contains all places of $K$ that ramify in $L$. For any character $\chi \in \widehat{G}$, we define the $L$-function

$$L_S(\chi, s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_S} \widetilde{\chi}(\mathfrak{a}) (\mathbf{N}\mathfrak{a})^{-s},$$

where $\widetilde{\chi}$ is defined by $\widetilde{\chi}(\mathfrak{p}) = \chi(\varphi_{\mathfrak{p}})$ for prime ideals, and extended to integral ideals by multiplicativity. Here $\varphi_{\mathfrak{p}}$ denotes the Frobenius element at $\mathfrak{p}$. We define modified $L$-functions in a manner analogous to the modified zeta functions. Specifically, if $T$ is finite and disjoint from $S$, we define

$$L_{S,T}(\chi, s) = L_S(\chi, s) \prod_{\mathfrak{q} \in T} \left(1 - \chi(\varphi_{\mathfrak{q}}) (\mathbf{N}\mathfrak{q})^{1-s}\right).$$

Next we introduce the *opposite Stickelberger element*, $\bar{\theta}_{S,T}$. This is the element of $\mathbb{C}[G]$ characterized by the property that

$$\chi(\bar{\theta}_{S,T}) = L_{S,T}(\chi, 0) \quad \text{for all characters } \chi \in \widehat{G}.$$

We use this terminology because it seems traditional to call the Stickelberger element the element that, under $\chi$, maps to the value of the $L$-function for the conjugate character, $\bar{\chi}$. There is also the opposite Stickelberger element $\bar{\theta}_S$, which is characterized by the property that

$$\chi(\bar{\theta}_S) = L_S(\chi, 0) \quad \text{for all characters } \chi \in \widehat{G}.$$

The element $\bar{\theta}_S$ nominally lives in $\mathbb{C}[G]$. However, it actually lives in $\mathbb{Q}[G]$; in the number field case, this is a consequence of a famous theorem of Seigel, in the function field case, it follows from Weil's work. Moreover, with our assumptions on $T$, $\bar{\theta}_{S,T}$ lies in $\mathbb{Z}[G]$. In the function field case, this again follows from Weil's work. In the number field case, it requires the $p$-adic congruences first proved by Barsky and Cassou-Noguès, and later by Deligne and Ribet.

Before introducing the Gross regulator, it is worthwhile to revisit the usual regulator. We order the places in $S$ as $v_0, v_1, \ldots, v_n$. Let $u_1, u_2, \ldots, u_n$ be a $\mathbb{Z}$-basis of the torsion-free group $U_{S,T}$, and consider the $(n+1) \times n$ matrix $(\log |u_j|_{v_i})$. We obtain the usual regulator by deleting an arbitrary row of this matrix, and taking the absolute value of the determinant. The Gross regulator is intended as an algebraic analogue, however, there is no algebraic analogue of the absolute value. For definiteness, we delete the 0-th row (corresponding to the place $v_0$). Then we can say that

$$\det \left(\log |u_j|_{v_i}\right)_{1 \le i,j, \le n} = \pm R_{S,T}$$

with the realization that the choice of sign depends upon the ordering of places in $S$, as well as the orientation of $\mathbb{Z}$-basis of $U_{S,T}$. We denote this determinant above as $\det_{\mathbb{R}}(\lambda_{S,T})$; then we have

$$\zeta_{S,T}(s) = \pm h_{S,T} \det\nolimits_{\mathbb{R}}(\lambda_{S,T}) s^n + O(s^{n+1}) \quad \text{near } s = 0.$$

Let $L/K$ be an abelian extension, unramified outside of $S$, with Galois group $G$. Consider the $(n+1) \times n$ matrix $(r_{v_i}(u_j))$, where $r_{v_i}$ is the local reciprocity map at the place $v_i$ for the extension $L/K$. Now $r_v(u_j) = 1$ for $v \notin S$, since $L/K$ is unramified outside of $S$, so the product formula tells us that the product of the elements in any given column is the identity element of $G$. Using the isomorphism $G \to I/I^2$ as an analogue of the logarithm, we define the Gross regulator to be

$$\det\nolimits_G(\lambda_{S,T}) = \det \left(r_{v_i}(u_j) - 1\right)_{1 \le i,j, \le n} \bmod I^{n+1}.$$

This definition also depends upon the ordering of places in $S$, as well as the orientation of $\mathbb{Z}$-basis of $U_{S,T}$, which may effect a change in sign. Otherwise, it is well-defined, modulo $I^{n+1}$.

Now we may state Gross' conjecture.

**Conjecture 1.3. (Gross' Conjecture)** *With the notation above, we have*

$$\bar{\theta}_{S,T} \equiv \pm h_{S,T} \det_G(\lambda_{S,T}) \bmod I^{n+1},$$

*where the sign is chosen to agree with the sign in*

$$\zeta_{S,T}(s) = \pm h_{S,T} \det_{\mathbb{R}}(\lambda_{S,T}) s^n + O(s^{n+1}) \quad \textit{near } s = 0.$$

The conjecture behaves well with respect to various functorialities, which we note here.

**Proposition 1.4.** *Suppose that Gross' conjecture holds for the extension $L/K$ with respect to $S$ and $T$. Let $L'/K$ be a subextension of $L/K$, $S'$ a superset of $S$ and $T'$ a superset of $T$.*

(a) *Gross' conjecture holds for $L'/K$ with respect to $S$ and $T$.*

(b) *Gross' conjecture holds for $L/K$ with respect to $S'$ and $T$.*

(c) *Gross' conjecture holds for $L/K$ with respect to $S$ and $T'$.*

Part (a) is more or less obvious from the functoriality of the Stickelberger element and the local reciprocity maps that comprise the Gross regulator. Parts (b) and (c) are also straightforward. We refer the reader to Aoki's paper [1] where the proofs are given in some detail.

As Gross notes, the functoriality of part (a) allows a formulation for infinite extensions by passing to the limit. The conjecture for an infinite extension is equivalent to it holding for every finite subextension.

**Proposition 1.5.** *Gross' conjecture holds for an extension $L/K$ if and only if it holds for each subextension of prime power degree.*

We will prove this proposition in the next section.

We enumerate here some cases of Gross' conjecture that have been proven.

**Theorem 1.6.** *Gross' conjecture holds in the following cases.*

(a) *$S$ is a singleton.*

(b) *$L/K$ is a constant field extension of function fields.*

(c) *$L/K$ is a $p$-power extension, or pro-$p$, and $K$ is a function field of characteristic $p$.*

(d) *$L/K$ is an elementary abelian $p$-extension, and the class number of $K$ is not divisible by $p$.*

(e) *$K = \mathbb{Q}$.*

(f) *$S$ contains a place that splits completely in $L$.*

**Proof.** Part (a) is immediate from the formulation of the conjecture. In this case, the conjecture becomes the usual class number formula.

For (b), the extension is unramified, so we may reduce $S$ to a singleton. Then it holds from part (a) and also 1.4(b).

Part (c) was proved by Tan [11].

Part (d) was proved by Lee [7].

Part (e) was proved by Aoki [1].

For part (f), the Stickelberger element is 0, because all the $L$-functions for $L/K$ vanish at $s = 0$. On the other hand, the Gross regulator vanishes modulo $I^{n+1}$, because the row of the matrix corresponding to the split place also vanishes. □

## Statement of result

Let $K = \mathbb{F}_q(X)$ be the rational function field over the finite field with $q$ elements. We will consider abelian extensions of $K$ that are unramified outside a set $S$ of three degree 1 places of $K$. Let $K_S$ be the maximal abelian extension of $K$ that is unramified outside of $S$.

**Theorem 1.7.** *If $T$ is any set of places, whose greatest common divisor of their degrees is relatively prime to $q-1$, then Gross' conjecture holds for $K_S/K$.*

**Remark.** The set $T$ should be considered secondary in the conjecture, so the restriction placed on it is only a minor issue. Nonetheless, it would be of some interest to eliminate this hypothesis.

## 2. Group rings

In this section we develop some general results about group rings and their augmentation ideals. Let $G$ be a group, $\mathbb{Z}[G]$ the group ring, and $I_G \subseteq \mathbb{Z}[G]$ its augmentation ideal. Our goal is a moderate understanding of congruences modulo $I_G^3$, so that we may verify the congruence of the conjecture. Of course, congruences modulo $I_G^2$ are well-understood, so we use that as our starting point.

In the following, when the group $G$ is understood, we will write simply $I$ rather than the more cumbersome $I_G$. The following lemma, whose proof we omit, is entirely elementary.

**Lemma 2.1.** (a) *If $a, b \in I$ and $a \equiv a' \bmod I^2$ and $b \equiv b' \bmod I^2$, then $ab \equiv a'b' \bmod I^3$.*
(b) *Suppose $G$ has exponent $n$. Then $n$ annihilates $I^r/I^{r+1}$ for every positive $r$.* □

We will use these two throughout the computation without mention.

**Lemma 2.2.** *Let $g \in G$ have order dividing $n$. If $n$ is odd, then $n(g-1) \equiv 0 \bmod I^3$, and if $n = 2m$ is even, then $n(g-1) \equiv m(g-1)^2 \bmod I^3$.*

**Proof.** We have $n(g-1) = [n-(1+g+g^2+\cdots+g^{n-1})](g-1) \equiv \frac{n(1-n)}{2}(g-1)^2 \bmod I^3$. If $n$ is odd, this is $0 \bmod I^3$, because $n(g-1) \in I^2$. If $n = 2m$ is even, then this is $m(g-1)^2 - mn(g-1)^2 \equiv m(g-1)^2 \bmod I^3$. □

**Lemma 2.3.** *Let $G$ be the direct product of two cyclic groups of order $n$. If $n$ is odd, then $\sum_{\gamma\in G}(\gamma-1) \equiv 0 \bmod I^3$. If $n$ is even, then $\sum_{\gamma\in G}(\gamma-1) \equiv (\sigma-1)^2 + (\sigma-1)(\tau-1) + (\tau-1)^2 \bmod I^3$, where $\sigma$ and $\tau$ generate the 2-torsion subgroup of $G$.*

**Proof.** Let $g, h$ be generators of the two cyclic components. Then

$$\sum_{\gamma\in G}(\gamma-1) = (1+g+g^2+\cdots+g^{n-1}-n)(1+h+h^2+\cdots+h^{n-1}-n) \\ + n(1+g+g^2+\cdots+g^{n-1}-n) + n(1+h+h^2+\cdots+h^{n-1}-n),$$

and note that

$$(1+g+g^2+\cdots+g^{n-1}-n)(1+h+h^2+\cdots+h^{n-1}-n) \\ \equiv \left(\tfrac{n(n-1)}{2}(g-1)\right)\left(\tfrac{n(n-1)}{2}(h-1)\right) \bmod I^3.$$

Also, $n$ kills $I^2/I^3$, so that $n(1+g+g^2+\cdots+g^{n-1}-n) \equiv n(\frac{n(n-1)}{2}(g-1)) \bmod I^3$. Moreover, $n(g-1) \equiv -\frac{n(n-1)}{2}(g-1)^2 \bmod I^3$, as in the proof of Lemma 2.1, and since $n(n-1)$ kills $I^2/I^3$, this is also $\equiv \frac{n(n-1)}{2}(g-1)^2 \bmod I^3$. Thus $n(1+g+g^2+\cdots+g^{n-1}-n) \equiv (\frac{n(n-1)}{2})^2(g-1)^2 \bmod I^3$, and similarly, $n(1+h+h^2+\cdots+h^{n-1}-n) \equiv (\frac{n(n-1)}{2})^2(h-1)^2 \bmod I^3$, so that

$$\sum_{\gamma\in G}(\gamma-1) \equiv \left(\tfrac{n(n-1)}{2}\right)^2((g-1)^2+(g-1)(h-1)+(h-1)^2) \bmod I^3.$$

If $n$ is odd, then $\left(\frac{n(n-1)}{2}\right)^2$ is divisible by $n$, and therefore kills $I^2/I^3$, so the sum in question vanishes modulo $I^3$.

On the other hand, if $n = 2m$ is even, then $\left(\frac{n(n-1)}{2}\right)^2 \equiv m^2 \bmod n$, so that

$$\sum_{\gamma\in G}(\gamma-1) \equiv m^2((g-1)^2+(g-1)(h-1)+(h-1)^2) \\ \equiv (g^m-1)^2+(g^m-1)(h^m-1)+(h^m-1)^2 \bmod I^3.$$

The proof concludes by noting that $g^m$ and $h^m$ generate the 2-torsion in $G$. $\square$

**Remark.** In the lemma, if $n$ is divisible by 4, then the sum $\sum_G(g-1) \equiv 0 \bmod I^3$, since $\left(\frac{n(n-1)}{2}\right)^2$ is divisible by $n$. However, we will not need this below.

Let $G_1$ [respectively $G_2$] be a finite group of exponent $n_1$ [respectively $n_2$], where $(n_1, n_2) = 1$, and let $G = G_1 \times G_2$. Let $\pi_i : G \to G_i$ be the natural projection and

$\iota_i : G_i \to G$ the inclusions defined by $\iota_1(g) = (g, 1)$ and $\iota_2(g) = (1, g)$. Let $I_G \subseteq \mathbb{Z}[G]$ denote the augmentation ideal, and similarly for $I_{G_i} \subseteq \mathbb{Z}[G_i]$.

**Lemma 2.4.** *If $\eta \in I_G$, then $\eta \equiv \iota_1(\pi_1(\eta)) + \iota_2(\pi_2(\eta)) \bmod I_G^r$ for any $r$.*

**Proof.** First consider an element of the form $\eta = g - 1$, for some $g \in G$, and let $g_i = \iota_i(\pi_i(g))$ for $i = 1, 2$, so that $g = g_1 g_2$. Then we have $\eta - \iota_1(\pi_1(\eta)) - \iota_2(\pi_2(\eta)) = (g_1 - 1)(g_2 - 1)$, so we must show that this element is in $I_G^r$. Since $n_1$ and $n_2$ are relatively prime, we may write $a_1 n_1 + a_2 n_2 = 1$ for some integers $a_1$ and $a_2$. Let

$$\theta = a_1(n_1 - (1 + g_1 + g_1^2 + \cdots + g_1^{n_1-1})) + a_2(n_2 - (1 + g_2 + g_2^2 + \cdots + g_2^{n_2-1})),$$

which is visibly an element of $I_G$. Since $g_i - 1$ annihilates $1 + g_i + g_i^2 + \cdots + g_i^{n_i-1}$, we have $\theta(g_1 - 1)(g_2 - 1) = (a_1 n_1 + a_2 n_2)(g_1 - 1)(g_2 - 1) = (g_1 - 1)(g_2 - 1)$. Thus, for any $r$, we have $(g_1 - 1)(g_2 - 1) = \theta^{r-2}(g_1 - 1)(g_2 - 1) \in I_G^r$, as desired. This proves the result for elements of the form $\eta = g - 1$. The result now holds for arbitrary $\eta \in I_G$ by $\mathbb{Z}$-linearity. $\square$

**Corollary 2.5.** *Let $\eta \in \mathbb{Z}[G]$. Then $\eta \in I_G^r$ if and only if $\pi_1(\eta) \in I_{G_1}^r$ and $\pi_2(\eta) \in I_{G_2}^r$.*

**Proof.** The "only if" part is trivial, and the "if" part is immediate for $r = 1$. Now suppose that $r > 1$, and that $\eta \in \mathbb{Z}[G]$ has the property that $\pi_i(\eta) \in I_{G_i}^r$ for $i = 1, 2$. Then $\eta \in I_G$, so from Lemma 1, $\eta \equiv \iota_1(\pi_1(\eta)) + \iota_2(\pi_2(\eta)) \bmod I_G^r$. Finally, $\iota_i(\pi_i(\eta)) \in \iota_i(I_{G_i}^r) \subseteq I_G^r$, for $i = 1, 2$, so that $\eta \in I_G^r$, as desired. $\square$

**Proof of Proposition 1.5.** This follows easily from Corollary 2.5 and the functoriality of both sides of the conjecture, as in 1.4(a). $\square$

## 3. Extensions ramified over three points of $\mathbb{P}^1$

Let $K = \mathbb{F}_q(X)$ be the rational function field over a finite field with $q$ elements, and $S = \{0, 1, \infty\}$ a set of three degree 1 places of $K$. Also let $p$ be the characteristic of $K$. Let $K_S$ be the maximal abelian extension of $K$ that is unramified outside of $S$. The Galois group $\mathrm{Gal}(K_S/K)$ factors canonically as the product of its prime-to-$p$ part and its $p$-power part. These correspond to linearly disjoint subextensions, $K_S^{(p)}/K$ and $K_S^{(\mathrm{not}\,p)}/K$. From Corollary 2.5, to verify Gross' conjecture for $K_S/K$, it is sufficient to verify it for $K_S^{(p)}/K$ and $K_S^{(\mathrm{not}\,p)}/K$. Tan's result (1.6(c)) handles the extension $K_S^{(p)}/K$. Thus our task is to verify Gross' conjecture for $K_S^{(\mathrm{not}\,p)}/K$; in fact, we will do slightly more. Let $K_S^{\mathrm{tame}}$ be the maximal abelian extension of $K$ that is unramified outside of $S$ and is only tamely ramified. Then clearly $K_S^{(\mathrm{not}\,p)} \subseteq K_S^{\mathrm{tame}}$. We will verify Gross' conjecture for $K_S^{\mathrm{tame}}/K$.

**Proposition 3.1.** $K_S^{\mathrm{tame}} = \overline{\mathbb{F}}_q(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$.

**Proof.** Let $J_K$ denote the idele group of $K$. We have a short exact sequence

$$1 \longrightarrow \prod_{v \in S} \kappa(v)^*/\text{image of}(\mathbb{F}_q^*) \longrightarrow J_K/(K^* \cdot \prod_{v \notin S} U_v \times \prod_{v \in S} U_v^1) \longrightarrow J_K/(K^* \cdot \prod_v U_v) \longrightarrow 1.$$

Moreover, $J_K/(K^* \cdot \prod_v U_v) = \mathrm{Pic}(K) \cong \mathbb{Z}$, since $\mathrm{Pic}^0(K)$ is trivial. The middle term of the exact sequence corresponds to the extension $K_S^{\mathrm{tame}}/K$, under classfield theory, so we get another short exact sequence

$$1 \longrightarrow \prod_{v \in S} \kappa(v)^*/\text{image of}(\mathbb{F}_q^*) \longrightarrow \mathrm{Gal}(K_S^{\mathrm{tame}}/K) \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 0,$$

where the quotient $\widehat{\mathbb{Z}}$ is the Galois group of the constant field extension $\overline{\mathbb{F}}_q(X)/K$. This sequence splits (non-canonically). It is easy to see that $\prod_{v \in S} \kappa(v)^*/\text{image of}(\mathbb{F}_q^*) \cong (\mathbb{F}_q^*)^2$, and then it is straightforward to identify $K(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$ as a complement. □

In order to prove Theorem 1.7, we will use the following calculation.

**Theorem 3.2.** *Let $L = \mathbb{F}_{q^m}(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$, $S = \{0, 1, \infty\}$ as above, and let $T$ contain a single place of degree $d$. Then Gross' conjecture holds for $L/K$ when both sides are multiplied by $(q^d - 1)/(q - 1)$, in other words, we have the congruence*

$$(1 + q + q^2 + \cdots + q^{d-1})\bar{\theta}_{S,T} \equiv (1 + q + q^2 + \cdots + q^{d-1}) h_{S,T} \mathrm{det}_G(\lambda_{S,T}) \bmod I^{n+1}.$$

Before proving Theorem 3.2, we will show how to prove Theorem 1.7 from it.

**Proof of Theorem 1.7.** As remarked above, we need only prove Gross' conjecture for $K_S^{\mathrm{tame}}/K$. Moreover, from 2.5, we may work one prime at a time.

Let $\ell$ be a prime. If $\ell$ does not divide $q - 1$, then the $\ell$-power part of $K_S^{\mathrm{tame}}/K$ is the constant field extension $\mathbb{F}_{q^{\ell^\infty}}(X)/K$, and Gross' conjecture is known for constant field extensions.

Now suppose that $\ell$ divides $q - 1$. By hypothesis, $T$ contains a place whose degree is not divisible by $\ell$. Let $\mathfrak{a} \in T$ be such a place, of degree $d$, and let $T_0 = \{\mathfrak{a}\}$. Also let $L_m = \mathbb{F}_{q^m}(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$. Theorem 3.2 shows that Gross' conjecture for $L_m/K$ holds when multiplied by $(1+q+q^2+\cdots+q^{d-1})$. The functoriality of the conjecture then implies that the same is true for any subextension of $L_m/K$, in particular, the $\ell$-power part of it, $L_m^{(\ell)}/K$. But then $\mathrm{Gal}(L_m^{(\ell)}/K)$ is an $\ell$-group, so $I^n/I^{n+1}$ is $\ell^\infty$-torsion. However, $1 + q + q^2 + \cdots + q^{d-1} \equiv d \not\equiv 0 \bmod \ell$, because $\ell$ divides $q - 1$, but not $d$. In particular, multiplication by $(1 + q + q^2 + \cdots + q^{d-1})$ is an automorphism of $I^n/I^{n+1}$. Therefore, the conjecture itself holds for $L_m^{(\ell)}/K$. Moreover, the compositum of $L_m^{(\ell)}$ over all $m$ is the $\ell$-power part of $K_S^{\mathrm{tame}}$, since the compositum of all such $L_m$ is $K_S^{\mathrm{tame}}$. Thus the conjecture holds for the $\ell$-power part of $K_S^{\mathrm{tame}}/K$, with respect to $T_0$, and therefore also with respect to $T$. This proves the theorem. □

Now we return to prove Theorem 3.2. Let $L = \mathbb{F}_{q^m}(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$, and $G = \mathrm{Gal}(L/K)$. Let $E_0 = \mathbb{F}_{q^m}(X)$ and $E_1 = \mathbb{F}_q(\sqrt[q-1]{X}, \sqrt[q-1]{X-1})$ be the given intermediate fields, and $G_i = \mathrm{Gal}(L/E_i)$, so that $G \cong G_0 \times G_1$.

The group $G_1$ is cyclic of order $m$, with a canonical generator, the "arithmetic" Frobenius, $F$, which acts on constants via the $q$-th power map.

Standard Kummer theory shows that $G_0 \cong (\mathbb{F}_q^*)^2$. For $\alpha, \beta \in \mathbb{F}_q^*$, let $\tau(\alpha,\beta) \in G_0$ be the element that acts via

$$\tau(\alpha,\beta)(\sqrt[q-1]{X}) = \alpha \sqrt[q-1]{X} \quad \text{and} \quad \tau(\alpha,\beta)(\sqrt[q-1]{X-1}) = \beta \sqrt[q-1]{X-1}$$

and acts trivially on constants. For notational convenience, we will let $\delta(\alpha,\beta)$ denote the group ring element $\tau(\alpha,\beta) - 1$.

For a place $v \notin S$ of $K$, let $\varphi_v$ denote its Frobenius element for the extension $L/K$. This is characterized by the condition that $\varphi_v(x) \equiv x^{\mathbf{N}v} \bmod v$ for all $x$ that are integral at $v$. We will only need to determine these Frobenius elements for degree 1 places.

**Lemma 3.3.** *Let $f(X) \in \mathbb{F}_q[X]$ be a monic irreducible polynomial of degree $d$. Then $f(X)$ divides $X^{1+q+q^2+\cdots+q^{d-1}} - (-1)^d f(0)$ and also $(X-1)^{1+q+q^2+\cdots+q^{d-1}} - (-1)^d f(1)$.*

**Proof.** Let $\alpha$ be a root of $f(X)$ in $\mathbb{F}_{q^d}$; then the other roots are $\alpha^q, \alpha^{q^2}, \alpha^{q^3}, \ldots, \alpha^{q^{d-1}}$. The product of these roots is $(-1)^d f(0)$. Therefore, over $\mathbb{F}_{q^d}$, $f(X)$ divides

$$X^{1+q+q^2+\cdots+q^{d-1}} - \alpha^{1+q+q^2+\cdots+q^{d-1}} = X^{1+q+q^2+\cdots+q^{d-1}} - (-1)^d f(0).$$

The same divisibility then holds in $\mathbb{F}_q[X]$, and the second divisibility follows in a similar way. $\square$

**Lemma 3.4.** *If $\mathfrak{a} \notin S$ is a degree $d$ place, which corresponds to a monic irreducible polynomial $f(X)$, its Frobenius element is given by $F^d\tau((-1)^d f(0), (-1)^d f(1))$. In particular, the Frobenius element of the degree 1 place at $b \neq 0,1$ is given by $F\tau(b, b-1)$.*

**Proof.** For a constant $\alpha \in \overline{\mathbb{F}}_q$, we have $\varphi_{\mathfrak{a}}(\alpha) \equiv \alpha^{\mathbf{N}\mathfrak{a}} = \alpha^{q^d} \bmod \mathfrak{a}$, and since $\varphi_{\mathfrak{a}}(\alpha)$ is also a constant, we have $\varphi_{\mathfrak{a}}(\alpha) = \alpha^{q^d}$. Furthermore,

$$\varphi_{\mathfrak{a}}(\sqrt[q-1]{X}) \equiv (X^{1+q+q^2+\cdots+q^{d-1}})(\sqrt[q-1]{X}) \equiv (-1)^d f(0)(\sqrt[q-1]{X}) \bmod \mathfrak{a}.$$

Since $\varphi_{\mathfrak{a}}(\sqrt[q-1]{X})$ is a constant times $\sqrt[q-1]{X}$, this congruence is in fact an equality. Similarly, $\varphi_{\mathfrak{a}}(\sqrt[q-1]{X-1}) = (-1)^d f(1)(\sqrt[q-1]{X-1})$. This proves the first statement, and the second is a special case. $\square$

Next we determine the Stickelberger element. Let $T$ be a singleton $\{\mathfrak{a}\}$ where $\mathfrak{a} \notin S$ is a place of degree $d$ which corresponds to the monic irreducible polynomial $f(X)$.

**Proposition 3.5.** $\bar{\theta}_{S,T} = (1 - q^d\varphi_{\mathfrak{a}})(1 + \sum_{b\neq 0,1} \varphi_b) + F^2(1 + qF + q^2F^2 + \cdots + q^{d-1}F^{d-1}) \sum_{g\in G_0} g.$

**Proof.** Let $\eta = (1 - q\varphi_{\mathfrak{a}})(1 + \sum_{b\neq 0,1} \varphi_b) + F^2(1 + qF + q^2F^2 + \cdots + q^{d-1}F^{d-1}) \sum_{g\in G_0} g$, and let $\chi$ be a character of $G$. We aim to show that $\chi(\eta) = L_{S,T}(\chi, 0)$. Note that $L_S(\chi, s) =$

$\sum_{n=0}^{\infty} q^{-ns} \sum_{\deg \mathfrak{b}=n} \widetilde{\chi}(\mathfrak{b})$, where the inner sum is over effective divisors of degree $n$ with support outside of $S$. In general, if $\chi$ is non-trivial on the subgroup corresponding to the maximal constant field extension, then this expression is a polynomial in $q^{-s}$, of degree $2g-2+\deg \mathfrak{f}$, where $g$ is the genus of the base field and $\mathfrak{f}$ is the conductor of $\chi$ (see [12, Chapter VII, Theorem 6]). In our situation, we have $g=0$ and $\deg \mathfrak{f}=3$, so the inner sum vanishes for $n \geq 2$. Therefore, if $\chi$ is non-trivial on $G_0$, then $L_S(\chi,s) = 1+q^{-s}\sum_{b\neq 0,1} \chi(\varphi_b)$. Now, still assuming that $\chi$ is non-trivial on $G_0$, we have

$$L_{S,T}(\chi,0) = (1-q\chi(\varphi_{\mathfrak{a}}))(1+\sum_{b\neq 0,1} \chi(\varphi_b)) = \chi(\eta),$$

since $\chi(\sum_{G_0} g) = 0$.

On the other hand, if $\chi$ is trivial on $G_0$, it is easy to calculate

$$L_{S,T}(\chi,s) = (1-\chi(F)q^{-s})^2 \left(1+\chi(F)q^{1-s} + (\chi(F)q^{1-s})^2 + \cdots + (\chi(F)q^{1-s})^{d-1}\right),$$

so that

$$L_{S,T}(\chi,0) = (1-\chi(F))^2(1+q\chi(F)+q^2\chi(F^2)+\cdots+q^{d-1}\chi(F^{d-1})).$$

Moreover, since $\chi$ is trivial on $G_0$, we have $\chi(\varphi_{\mathfrak{a}}) = \chi(F^d)$ and $\chi(\varphi_b) = \chi(F)$, so that

$$\begin{aligned}\chi(\eta) = (1-q^d\chi(F^d))(1+(q-2)\chi(F)) \\ + (q-1)^2\chi(F^2)((1+q\chi(F)+q^2\chi(F^2)+\cdots+q^{d-1}\chi(F^{d-1})),\end{aligned}$$

which equals the value of $L_{S,T}(\chi,0)$ above. This shows that $\chi(\eta) = L_{S,T}(\chi,0)$ for all $\chi \in \widehat{G}$, so $\eta = \bar{\theta}_{S,T}$, as claimed. $\square$

Now we attempt to compute the Gross regulator. In general, we do not know enough units to compute it exactly, but we can compute some multiple of it. The unit group $U_S$ is generated by $\mathbb{F}_q^*$, $X$ and $X-1$. We easily identify two independent units in $U_{S,T}$, namely, $u_0 = X^{1+q+q^2+\cdots+q^{d-1}}/((-1)^d f(0))$ and $u_1 = (X-1)^{1+q+q^2+\cdots+q^{d-1}}/((-1)^d f(1))$. For notational convenience, we let $\alpha$ denote $(-1)^d f(0)$ and $\beta = (-1)^d f(1)$. Let $V$ be the group generated by $u_0$ and $u_1$. The index $(U_S : V)$ is $(1+q+q^2+\cdots+q^{d-1})(q^d-1)$, whereas $(U_S : U_{S,T}) = (q^d-1)/h_{S,T}$. Therefore $(U_{S,T} : V) = (1+q+q^2+\cdots+q^{d-1})h_{S,T}$, whence

$$(1+q+q^2+\cdots+q^{d-1})h_{S,T}\det_G(\lambda_{S,T}) \equiv \det\begin{pmatrix} r_0(u_0)-1 & r_0(u_1)-1 \\ r_1(u_0)-1 & r_1(u_1)-1\end{pmatrix} \bmod I^3, \quad (3.6)$$

where $r_0$ and $r_1$ are the local reciprocity maps at 0 and 1 respectively.

**Proposition 3.7.** *We have the following values of the local reciprocity maps:*

$$\begin{aligned} r_0(u_0) &= F^{-(1+q+q^2+\cdots+q^{d-1})}\tau((-1)^d\alpha^{-1},(-1)^d) \\ r_0(u_1) &= \tau((-1)^d\beta^{-1},1) \\ r_1(u_0) &= \tau(1,\alpha^{-1}) \\ r_1(u_1) &= F^{-(1+q+q^2+\cdots+q^{d-1})}\tau(1,(-1)^d\beta^{-1}) \end{aligned}$$

**Proof.** On constants, $r_0(u_0)$ acts as $F^{-\operatorname{ord}_0(u_0)} = F^{-(1+q+q^2+\cdots+q^{d-1})}$. For its action on $\sqrt[q-1]{X}$ and $\sqrt[q-1]{X-1}$, we use the "tame symbol" (see Serre [9, Chapter XIV, Proposition 8]). We calculate

$$(u_0,X)_0 = (-1)^{1+q+q^2+\cdots+q^{d-1}}u_0/X^{1+q+q^2+\cdots+q^{d-1}} \equiv (-1)^d\alpha^{-1} \bmod \mathfrak{p}_0$$

and

$$(u_0,X-1)_0 = (-1)^0u_0^0/(X-1)^{1+q+q^2+\cdots+q^{d-1}} \equiv (-1)^d \bmod \mathfrak{p}_0.$$

Thus $r_0(u_0) = F^{-(1+q+q^2+\cdots+q^{d-1})}\tau((-1)^d\alpha^{-1},(-1)^d)$, as claimed. The other values are similar computations. □

In order to simplify (3.6) above, we need a lemma.

**Lemma 3.8.** *For $x,y\in\mathbb{F}_q^*$, we have $\delta(x,1)\delta(1,y)\equiv\delta(y,1)\delta(1,x) \bmod I^3$.*

**Proof.** We use the fact that $\mathbb{F}_q^*$ is cyclic; let $t$ be a generator. Then $t^m = x$ and $t^n = y$ for some integers $m,n$. Then we have $\delta(x,1)\delta(1,y)\equiv m\delta(t,1)n\delta(1,t) \bmod I^3$, and similarly, $\delta(y,1)\delta(1,x)\equiv n\delta(t,1)m\delta(1,t) \bmod I^3$. □

**Proposition 3.9.** *We have*

$$\begin{aligned} (1+q+q^2+\cdots+q^{d-1})h_{S,T}\det_G(\lambda_{S,T}) &\equiv (1+q+q^2+\cdots+q^{d-1})^2(F-1)^2 \\ &\quad + (1+q+q^2+\cdots+q^{d-1})(F-1)\delta((-1)^d\alpha,\beta) \\ &\quad + d\delta(-1,-1)\delta(1,(-1)^d\beta) \bmod I^3. \end{aligned}$$

**Proof.** From 3.7, we have

$$r_0(u_0)-1\equiv-(1+q+q^2+\cdots+q^{d-1})(F-1)-\delta((-1)^d\alpha,(-1)^d) \bmod I^2,$$

as well as $r_0(u_1)-1\equiv-\delta((-1)^d\beta,1) \bmod I^2$, $r_1(u_0)-1\equiv-\delta(1,\alpha) \bmod I^2$, and

$$r_1(u_1)-1\equiv-(1+q+q^2+\cdots+q^{d-1})(F-1)-\delta(1,(-1)^d\beta) \bmod I^2.$$

Now it follows that

$$\begin{aligned}
&(1+q+q^2+\cdots+q^{d-1})h_{S,T}\det_G(\lambda_{S,T})\\
&\qquad\equiv (1+q+q^2+\cdots+q^{d-1})^2(F-1)^2\\
&\qquad\quad + (1+q+q^2+\cdots+q^{d-1})(F-1)[\delta((-1)^d\alpha,(-1)^d)+\delta(1,(-1)^d\beta)]\\
&\qquad\qquad + \delta((-1)^d\alpha,(-1)^d)\delta(1,(-1)^d\beta) - \delta((-1)^d\beta,1)\delta(1,\alpha) \bmod I^3,
\end{aligned}$$

from equation (3.6) above. We also have

$$(F-1)[\delta((-1)^d\alpha,(-1)^d)+\delta(1,(-1)^d\beta)] \equiv (F-1)\delta((-1)^d\alpha,\beta) \bmod I^3,$$

and, using the lemma,

$$\begin{aligned}
\delta((-1)^d\alpha,(-1)^d)\delta(1,(-1)^d\beta) &\equiv [\delta((-1)^d,(-1)^d)+\delta(\alpha,1)]\delta(1,(-1)^d\beta)\\
&\equiv d\delta(-1,-1)\delta(1,(-1)^d\beta) + \delta((-1)^d\beta,1)\delta(1,\alpha) \bmod I^3,
\end{aligned}$$

from which the result follows. ☐

We now return to consider the Stickelberger element. To simplify the expression for this, we need a few easy lemmas.

**Lemma 3.10.** *We have* $\sum_{g\in G_0}(g-1) \equiv \delta(-1,1)^2 + \delta(1,-1)\delta(-1,-1) \bmod I^3$.

**Proof.** If $q$ is even, then the left hand side vanishes modulo $I^3$, from Lemma 2.3, while the right hand side vanishes because $-1 = 1$. If $q$ is odd, then it also follows from Lemma 2.3, by noting that $\tau(-1,1)$ and $\tau(1,-1)$ generate the 2-torsion in $G_0$, and also that $\delta(-1,1)\delta(1,-1) + \delta(1,-1)^2 \equiv \delta(1,-1)\delta(-1,-1) \bmod I^3$. ☐

**Lemma 3.11.** *For any* $x \in \mathbb{F}_q^*$, *we have* $(q-1)\delta(x,1) \equiv \delta(x,1)\delta(-1,1) \bmod I^3$, *and* $(q-1)\delta(1,x) \equiv \delta(1,x)\delta(1,-1) \bmod I^3$.

**Proof.** We prove the first statement; the second is similar. If $q$ is even, then the left hand side vanishes modulo $I^3$ by Lemma 2.2, while the right hand side vanishes because $-1 = 1$. If $q$ is odd, then Lemma 2.2 shows that $(q-1)\delta(x,1) \equiv \frac{q-1}{2}\delta(x,1)^2 \equiv \delta(x,1)\delta(x^{(q-1)/2},1) \bmod I^3$. Thus we must show that $\delta(x,1)\delta(-x^{(q-1)/2},1) \equiv 0 \bmod I^3$. If $x$ is a square in $\mathbb{F}_q$, say $x = y^2$, then $\delta(x,1)\delta(-x^{(q-1)/2},1) = \delta(y^2,1)\delta(-1,1) \equiv 2\delta(y,1)\delta(-1,1) \equiv \delta(y,1)\delta(1,1) = 0 \bmod I^3$. On the other hand, if $x$ is not a square, then $\delta(-x^{(q-1)/2},1) = \delta(1,1) = 0$. ☐

Now we return our attention to the Stickelberger element.

**Proposition 3.12.** *We have the congruence*

$$\bar{\theta}_{S,T} \equiv (1+q+q^2+\cdots+q^{d-1})(F-1)^2 + (F-1)\delta((-1)^d\alpha,\beta) + \delta(\alpha,\beta)\delta(-1,-1) \bmod I^3.$$

**Proof.** We begin with the expression in Proposition 3.5, and write

$$\begin{aligned}(1-q^d\varphi_{\mathfrak{a}})(1+\sum_{b\neq 0,1}\varphi_b)\\ &= (q-1)(1-q^d) - q^d(q-1)(\varphi_{\mathfrak{a}}-1) + (1-q^d)\sum_{b\neq 0,1}(\varphi_b-1)\\ &\qquad - q^d(\varphi_{\mathfrak{a}}-1)\sum_{b\neq 0,1}(\varphi_b-1)\end{aligned}$$

and now consider each term individually. The term $(q-1)(1-q^d)$ does not require any simplification.

Firstly, we have

$$\begin{aligned}-q^d(q-1)(\varphi_{\mathfrak{a}}-1) &= -q^d(q-1)(F^d\tau(\alpha,\beta)-1)\\ &= -q^d(q-1)((F^d-1)\delta(\alpha,\beta)+(F^d-1)\\ &\qquad + \delta(\alpha,1)\delta(1,\beta)+\delta(\alpha,1)+\delta(1,\beta))\\ &\equiv -q^d(q-1)((F^d-1)+\delta(\alpha,1)+\delta(1,\beta)) \bmod I^3,\end{aligned}$$

because $(q-1)$ kills $\delta(\alpha,\beta),\delta(\alpha,1)$ and $\delta(1,\beta)$, modulo $I^2$. Also, from 3.11, we have

$$-q^d(q-1)\delta(\alpha,1) \equiv -q^d\delta(\alpha,1)\delta(-1,1) \equiv -\delta(\alpha,1)\delta(-1,1) \equiv \delta(\alpha,1)\delta(-1,1) \bmod I^3$$

and similarly, $-q^d(q-1)\delta(1,\beta)\equiv \delta(1,\beta)\delta(1,-1) \bmod I^3$. Expand $(F^d-1)$ in powers of $(F-1)$ to see that $(F^d-1)\equiv d(F-1)+\frac{d(d-1)}{2}(F-1)^2 \bmod I^3$. Therefore we get

$$\begin{aligned}-q^d(q-1)(\varphi_{\mathfrak{a}}-1) &\equiv -dq^d(q-1)(F-1) - \tfrac{d(d-1)}{2}q^d(q-1)(F-1)^2\\ &\qquad + \delta(\alpha,1)\delta(-1,1)+\delta(1,\beta)\delta(1,-1) \bmod I^3 \qquad (3.13)\end{aligned}$$

Secondly, we have

$$\begin{aligned}(1-q^d)\sum_{b\neq 0,1}(\varphi_b-1) &= (1-q^d)\sum_{b\neq 0,1}((F-1)\delta(b,b-1)+(F-1)+\delta(b,b-1))\\ &\equiv (q-2)(1-q^d)(F-1)+(1-q^d)\sum_{b\neq 0,1}\delta(b,b-1),\end{aligned}$$

because $(q-1)\delta(b,b-1)\in I^2$. Moreover, we have $\sum_{b\neq 0,1}\delta(b,b-1)\equiv\delta(-1,1) \bmod I^2$, from Wilson's Theorem, and because $(q-1)$ kills $I^2_{G_0}/I^3_{G_0}$, we have

$$\begin{aligned}(1-q^d)\sum_{b\neq 0,1}\delta(b,b-1) &\equiv (1-q^d)\delta(-1,1)\\ &\equiv -(1+q+q^2+\cdots+q^{d-1})\delta(-1,1)^2\\ &\equiv d\delta(-1,1)^2 \bmod I^3,\end{aligned}$$

also using 3.11. Therefore we have

$$(1-q^d)\sum_{b\neq 0,1}(\varphi_b-1)\equiv (q-2)(1-q^d)(F-1)+d\delta(-1,1)^2 \bmod I^3. \tag{3.14}$$

Thirdly, we have

$$\begin{aligned}(\varphi_{\mathfrak{a}}-1)\sum_{b\neq 0,1}(\varphi_b-1) &\equiv [d(F-1)+\delta(\alpha,\beta)]\sum_{b\neq 0,1}[(F-1)+\delta(b,b-1)]\\ &\equiv [d(F-1)+\delta(\alpha,\beta)][(q-2)(F-1)+\delta(-1,1)]\\ &\equiv d(q-2)(F-1)^2-(F-1)\delta((-1)^d\alpha,\beta)\\ &\qquad\qquad\qquad\qquad +\delta(\alpha,\beta)\delta(-1,1) \bmod I^3.\end{aligned}$$

Multiply by $-q^d$ to get

$$\begin{aligned}-q^d(\varphi_{\mathfrak{a}}-1)\sum_{b\neq 0,1}(\varphi_b-1) &\equiv -dq^d(q-2)(F-1)^2+(F-1)\delta((-1)^d\alpha,\beta)\\ &\qquad\qquad\qquad +\delta(\alpha,\beta)\delta(-1,1) \bmod I^3. \end{aligned}\tag{3.15}$$

Lastly, since $\sum_{g\in G_0} g\equiv (q-1)^2+\delta(-1,1)^2+\delta(1,-1)\delta(-1,-1) \bmod I^3$, we get

$$\begin{aligned}&F^2(1+qF+q^2F^2+\cdots+q^{d-1}F^{d-1})\sum_{g\in G_0} g\\ &\qquad\equiv (q-1)^2F^2(1+qF+q^2F^2+\cdots+q^{d-1}F^{d-1})\\ &\qquad\qquad +(1+q+q^2+\cdots+q^{d-1})(\delta(-1,1)^2+\delta(1,-1)\delta(-1,-1)) \bmod I^3.\end{aligned}$$

We also have

$$\begin{aligned}&(1+q+q^2+\cdots+q^{d-1})(\delta(-1,1)^2+\delta(1,-1)\delta(-1,-1))\\ &\qquad\qquad\qquad \equiv d(\delta(-1,1)^2+\delta(1,-1)\delta(-1,-1)) \bmod I^3.\end{aligned}$$

Expand $(q-1)^2F^2(1+qF+q^2F^2+\cdots+q^{d-1}F^{d-1})$ in ascending powers of $(F-1)$ to see that it is congruent to $c_0+c_1(F-1)+c_2(F-1)^2$ modulo $I^3$, where the coefficients are $c_0=(q-1)(q^d-1)$, $c_1=(d+1)q^{d+1}-(d+2)q^d-q+2$, and

$$c_2=1+q+q^2+\cdots+q^{d-1}-\tfrac{d(d+3)}{2}q^d+\tfrac{d(d+1)}{2}q^{d+1}.$$

Putting these together, we obtain

$$\begin{aligned}&F^2(1+qF+q^2F^2+\cdots+q^{d-1}F^{d-1})\sum_{g\in G_0} g\\ &\qquad\equiv c_0+c_1(F-1)+c_2(F-1)^2+d(\delta(-1,1)^2+\delta(1,-1)\delta(-1,-1)) \bmod I^3.\end{aligned}\tag{3.16}$$

Now add together equations (3.13) through (3.16) to obtain

$$\bar{\theta}_{S,T} \equiv (1+q+q^2+\cdots+q^{d-1})(F-1)^2 + (F-1)\delta((-1)^d\alpha,\beta) + \delta(\alpha,\beta)\delta(-1,1) \\ + \delta(\alpha,1)\delta(-1,1) + \delta(1,\beta)\delta(1,-1) + d\delta(1,-1)\delta(-1,-1).$$

Finally, we have

$$\begin{aligned} \delta(\alpha,\beta)\delta(-1,1) + \delta(\alpha,1)\delta(-1,1) + \delta(1,\beta)\delta(1,-1) + d\delta(1,-1)\delta(-1,-1) \\ &\equiv \delta(1,\beta)\delta(-1,1) + \delta(1,\beta)\delta(1,-1) + d\delta(1,-1)\delta(-1,-1) \\ &\equiv \delta(1,\beta)\delta(-1,-1) + d\delta(1,-1)\delta(-1,-1) \\ &\equiv \delta(1,(-1)^d\beta)\delta(-1,-1) \bmod I^3, \end{aligned}$$

which proves the required congruence. □

Finally, we complete the proof of Theorem 3.2.

**Proof of Theorem 3.2.** Start with the congruence of Proposition 3.12, and multiply by $(1+q+q^2+\cdots+q^{d-1})$. Note also that

$$(1+q+q^2+\cdots+q^{d-1})\delta(1,(-1)^d\beta)\delta(-1,-1) \equiv d\delta(1,(-1)^d\beta)\delta(-1,-1) \bmod I^3,$$

because $(q-1)$ kills $I_{G_0}^2/I_{G_0}^3$. Now the theorem follows by comparing with the congruence in Proposition 3.9. □

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