Published in *L'Enseignement Mathématique* **49** (2003), no. 1–2, pp. 123–155.

TILE HOMOTOPY GROUPS

Michael Reid

University of Central Florida

September 17, 2003

Abstract. The technique of using checkerboard colorings to show impossibility of some tiling problems is well-known. Conway and Lagarias have introduced a new technique using boundary words. They show that their method is at least as strong as any generalized coloring argument. They successfully apply their technique, which involves some understanding of specific finitely presented groups, to two tiling problems. Partly because of the difficulty in working with finitely presented groups, their technique has only been applied in a handful of cases.

We present a slightly different approach to the Conway-Lagarias technique, which we hope provides further insight. We also give a strategy for working with the finitely presented groups that arise, and we are able to apply it in a number of cases.

1. Introduction

A classical problem is the following (see [3, pp. 142, 394], [7]).

Remove two diagonally opposite corners from a checkerboard. Dominoes are placed on the board, each covering exactly two (vertically or horizontally) adjacent squares. Can all 62 squares be covered by 31 dominoes?

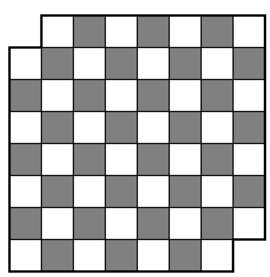


Figure 1.1. Mutilated checkerboard

2000 *Mathematics Subject Classification.* 52C20 (05B45 20F32).

Typeset by $\mathcal{AMS}$-TEX

The key to the solution is to note that each domino covers one black square and one red square, whereas the "mutilated checkerboard" has 32 squares of one color and 30 of the other color. Therefore we see that it cannot be tiled.

A smaller version of this problem uses a mutilated 4×4 checkerboard. For this problem, exhaustive analysis is easy; there are two ways to cover the marked square. In the first case, this forces the location of the next 3 dominoes, and isolates a square that cannot be covered. In the second case, the next 5 dominoes are forced, again isolating a square that cannot be covered.

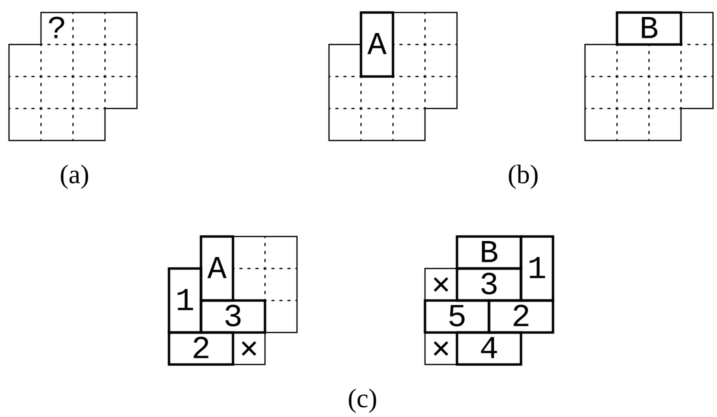
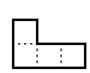


Figure 1.2. Analysis of mutilated 4×4 checkerboard. (a) First cell to cover. (b) Two ways to cover it. (c) Both cases force a contradiction.

A similar exhaustive analysis can be applied to the mutilated 8×8 checkerboard, but it is dramatically more cumbersome. The elegance of this approach may be questionable, but its validity is fine.

This is the type of problem we will consider in this paper. We will have a finite set $\mathcal{T}$ of polyomino prototiles, and a finite region we are trying to tile with $\mathcal{T}$. There is no restriction on the use of tiles in $\mathcal{T}$; we may use any tile repeatedly, or we may fail to utilize any given tile. We will be interested in negative results, where we can show that the region cannot be tiled. In light of the remarks above concerning exhaustive search, we will be especially interested in techniques that can prove that infinitely many such regions are untileable. (Although the example of the mutilated checkerboard is only a single shape, it is clear that the same technique applies to infinitely many regions.)

To fix ideas, we will mainly focus on the following type of tiling problem. Our protoset will be a small set of polyominoes, and we'll be interested in tiling rectangles with the set. The same techniques work with little modification for protosets consisting of "polyiamonds" or "polyhexes".

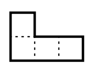
Another typical example is the following. Can 25 copies of the shape  cover a 10×10 square? (The tiles may be rotated and/or reflected.) Again, the answer is "no".

Label the squares in alternate rows by 1 and 5, as shown.

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5

Figure 1.3. 10×10 square.

Then every placement of a tile covers either one 1 and three 5's, or one 5 and three 1's. In either case, the total it covers is a multiple of 8. However, the 10×10 square covers a total of 300, which is not a multiple of 8, so the square cannot be tiled.

Although the 10×10 square is a single shape, and thus can be exhaustively examined, this same numbering argument shows that  cannot tile any rectangle whose area is congruent to 4 modulo 8. See [8] [10] [11, pp. 42–43] for this example. We will show below (Proposition 2.10) that this type of argument can always be done by a suitable numbering of the squares.

2. Tiling and Integer Programming

Here we translate a polyomino tiling problem into an algebra question. Consider, for example, the problem of tiling the fairly simple shape

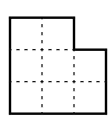


Figure 2.1. Region to tile with dominoes.

by dominoes. For each possible tile placement, we introduce a variable, x_i, which indicates how many times that placement occurs in the tiling.

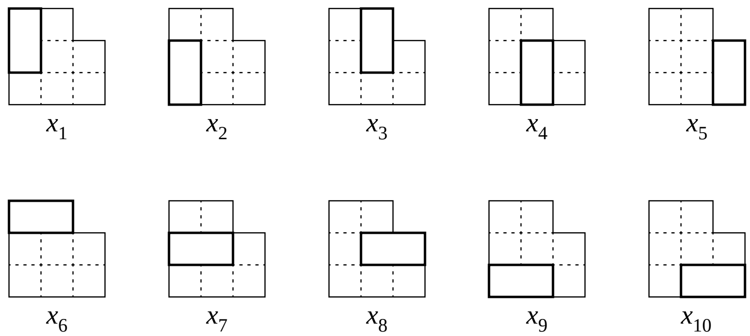


Figure 2.2. Possible tile placements and associated variables.

In particular, its value will be either 0 or 1. Each cell of the region gives a linear equation, which indicates that the cell is covered exactly once. Thus, for the example of Figure 2.1, we get the system of linear equations

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad + x_6 \qquad\qquad\qquad\qquad &= 1 \\
x_3 \qquad\quad + x_6 \qquad\qquad\qquad\qquad &= 1 \\
x_1 + x_2 \qquad\qquad\qquad\qquad + x_7 \qquad\qquad\qquad &= 1 \\
x_3 + x_4 \qquad\quad + x_7 + x_8 \qquad\qquad &= 1 \\
x_5 \qquad\qquad + x_8 \qquad\qquad &= 1 \\
x_2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad + x_9 \qquad &= 1 \\
x_4 \qquad\qquad\qquad\qquad + x_9 + x_{10} &= 1 \\
x_5 \qquad\qquad\qquad\qquad + x_{10} &= 1
\end{aligned} \tag{2.3}$$

A tiling then corresponds to a solution to the system above. However, the converse is not true; as noted above, the value of each variable must be either 0 or 1. A solution to the system in which every variable takes the value 0 or 1 indeed corresponds to a tiling.

Instead of making this requirement on the variables, it is sufficient (and perhaps more natural) to insist only that the values are non-negative integers. A linear system, such as (2.3) above, in which the coefficients are non-negative integers, where we seek solutions in non-negative integers, is one form of the *integer programming problem*. It is known that the general integer programming problem is NP-complete, see [16]. It has also been shown that the general problem of tiling a finite region by a set of polyominoes is NP-complete, see [6] [12].

Linear Algebra and Signed Tilings

If we relax the condition that the variables take non-negative values, we have a more tractable, although somewhat different problem. Indeed, it is simply a linear algebra problem, albeit over $\mathbb{Z}$, but its resolution by row-reduction is straightforward.

A solution to (2.3) in integers, possibly negative, corresponds to a "signed tiling", i.e. where tiles may be subtracted from the region. Equivalently, we can think of allowing "anti-tiles". Again however, we do not quite have a one-to-one correspondence, because a signed tiling may utilize cells outside the region. Thus it is appropriate to consider all the cells of the square lattice when considering signed tilings.

Tile Homology Groups

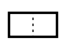
Following Conway and Lagarias, we define the tile homology group of a protoset $\mathcal{T}$. Let A be the free abelian group on all the cells of the square lattice. To a placement of a tile in $\mathcal{T}$, we associate the element of A which is 1 in those coordinates whose cell is covered by the tile placement, and is 0 in all other coordinates. Note that this element depends upon the particular placement of the tile. In the same way, to a region, we also associate an element of A. Again, this element depends upon the location and orientation of the region. For simplicity, we will consider a region to be a *fixed* subset of the square lattice.

2.4. Definition. The *tile homology group* of $\mathcal{T}$ is the quotient $H(\mathcal{T}) = A/B(\mathcal{T})$, where $B(\mathcal{T}) \subseteq A$ is the subgroup generated by all elements corresponding to possible placements of tiles in $\mathcal{T}$.

The relevance of the tile homology group is clear. A region R has a tiling by $\mathcal{T}$ if and only if the element corresponding to R is in the submonoid of A generated by elements corresponding to tile placements, and it has a signed tiling if and only if the corresponding element is in $B(\mathcal{T})$. Thus $H(\mathcal{T})$ measures the obstruction to having a signed tiling by $\mathcal{T}$.

We introduce some conventions that will be useful. The cell with lower left corner at the point (i, j) we be called simply the (i, j) cell. We let a_{ij} denote the element of A corresponding to this cell, and let $\bar{a}_{ij}$ denote its image in $H(\mathcal{T})$.

The tile homology group is defined by infinitely many generators and infinitely many relations. In this form, it is somewhat difficult to use. In a number of simple cases, we can show that it is finitely generated.

2.5. Example. $\mathcal{T} = \{$  $\}$, both orientations allowed. $H(\mathcal{T})$ is defined by

$$\begin{array}{lll}
\text{Generators:} & \bar{a}_{ij} & i, j \in \mathbb{Z} \\
\text{Relations:} & \bar{a}_{ij} + \bar{a}_{i+1,j} = 0 & i, j \in \mathbb{Z} \\
 & \bar{a}_{ij} + \bar{a}_{i,j+1} = 0 & i, j \in \mathbb{Z}
\end{array}$$

Note that we have

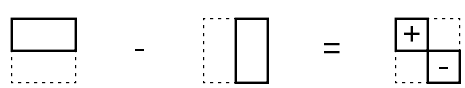
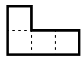


Figure 2.6. Translation of a square by 1 diagonal unit.

which shows that $\bar{a}_{ij} - \bar{a}_{i+1,j-1} = 0$ in $H(\mathcal{T})$. Similarly, by rotating this figure by 90 degrees, we obtain $\bar{a}_{ij} - \bar{a}_{i+1,j+1} = 0$. These show that $H(\mathcal{T})$ is generated by the two elements $\bar{a}_{00}$ and $\bar{a}_{01}$. Now the relations above collapse into a single relation between these two generators: $\bar{a}_{00} + \bar{a}_{01} = 0$. Thus we see that $H(\mathcal{T}) \cong \mathbb{Z}$, and a specific isomorphism is given by $[R] \mapsto (b - r)$, where the region R has b black squares and r red squares. This shows that a region has a signed tiling by dominoes if and only if it has the same number of black squares as it has red squares.

2.7. Example. $\mathcal{T} = \{$  $\}$, all rotations and reflections allowed. From the equation

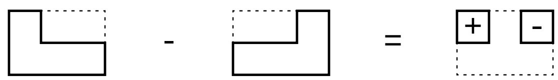


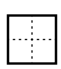
Figure 2.8. Translation of a square by 2 units.

we see that $\bar{a}_{ij} = \bar{a}_{i+2,j}$, and similarly, we have $\bar{a}_{ij} = \bar{a}_{i,j+2}$. Thus $H(\mathcal{T})$ is generated by $\bar{a}_{00}, \bar{a}_{01}, \bar{a}_{10}$ and $\bar{a}_{11}$. The relations become

$$\begin{aligned}
2\bar{a}_{00} + \bar{a}_{01} + \bar{a}_{10} \phantom{{}+ \bar{a}_{11}} &= 0 \\
\bar{a}_{00} + 2\bar{a}_{01} + \bar{a}_{11} &= 0 \\
\bar{a}_{00} + 2\bar{a}_{10} + \bar{a}_{11} &= 0 \\
\bar{a}_{01} + \bar{a}_{10} + 2\bar{a}_{11} &= 0
\end{aligned}$$

so we easily find that $H(\mathcal{T}) \cong \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. A specific isomorphism is given by $[R] \mapsto (A - B - C + D, (2A + B - C) \bmod 4)$, where the region R contains A [respectively, B, C, D] (i, j) cells with i and j both even [respectively, i even and j odd, i odd and j even, i and j both odd]. From this analysis, we can easily find the numbering used in Figure 1.3 above.

In general, the tile homology group will not be finitely generated. A simple example that illustrates this is the following.

2.9. Example. Let $\mathcal{T} = \{$  $\}$. It is easy to show that $H(\mathcal{T})$ is a free abelian group on the generators which are images of all (i, j) cells with $i = 0$ or $j = 0$. In particular, $H(\mathcal{T})$ is not finitely generated.

We now show that the types of proofs given in the examples of the introduction can always be given using a suitable numbering of the cells of the square lattice.

2.10. Proposition. *Let R be a region that does not have a signed tiling by the protoset $\mathcal{T}$. Then there is a numbering of all the cells in the square lattice with rational numbers such that*

(1) Any placement of a tile covers a total that is an integer, and
(2) The total covered by the region is not an integer.

Proof. Let $r \in H(\mathcal{T})$ be the image of the region R in the tile homology group, which by hypothesis, is non-trivial. Let $\langle r \rangle \subseteq H(\mathcal{T})$ be the cyclic subgroup generated by r. Note that there is a homomorphism $\varphi : \langle r\rangle \to \mathbb{Q}/\mathbb{Z}$ with $\varphi(r) \neq 0$. For example, if r has infinite order, then φ may be defined by $\varphi(r) = \frac{1}{2} \bmod \mathbb{Z}$, while if r has finite order, $n > 1$, then we may take $\varphi(r) = \frac{1}{n} \bmod \mathbb{Z}$. Now $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, so the homomorphism φ extends to a homomorphism $H(\mathcal{T}) \to \mathbb{Q}/\mathbb{Z}$, also called φ, which is defined on all of $H(\mathcal{T})$. Since A is a free abelian group, the composite map $A \twoheadrightarrow A/B(\mathcal{T}) = H(\mathcal{T}) \xrightarrow{\varphi} \mathbb{Q}/\mathbb{Z}$ lifts to a homomorphism $\psi : A \to \mathbb{Q}$, such that the following square commutes

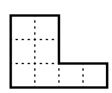
$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d] & \mathbb{Q} \arrow[d] \\ H(\mathcal{T}) \arrow[r, "\varphi"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

where the vertical arrows are the natural projections. Then ψ defines a numbering of the squares with rational numbers. Moreover, $B(\mathcal{T})$ is in the kernel of $A \to \mathbb{Q}/\mathbb{Z}$, which means that every tile placement covers an integral total. Also, R covers a total that is not an integer, because $\varphi(r) \neq 0$. $\square$

2.11. Remark. In many cases that we have examined, $H(\mathcal{T})$ is finitely generated, so that $\varphi(H(\mathcal{T})) \subseteq \mathbb{Q}/\mathbb{Z}$ is also finitely generated, whence $\varphi(H(\mathcal{T})) \subseteq \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ for some integer N. In such cases, it seems convenient to clear denominators by multiplying everything by N. We thus obtain a numbering of the squares by integers, such that

(1′) Any placement of a tile covers a total divisible by N, and
(2′) The region covers a total that is not divisible by N.

This shows that generalized checkerboard coloring arguments such as in [10, Thm. 6] can be given in a simpler form. We provide a numbering proof of Klarner's result, which is based upon his coloring.

2.12. Proposition. *Let $\mathcal{T} = \{$*  *$\}$, with all orientations allowed. If $\mathcal{T}$ tiles a rectangle, then its area is divisible by* 16.

Proof. We must show that $\mathcal{T}$ cannot tile a $(2m + 1) \times (16n + 8)$ rectangle or a $(4m + 2) \times (8n + 4)$ rectangle. Number the squares by

$$(i, j) \mapsto \begin{cases} 5 & \text{if } i \equiv 0 \bmod 4, \\ -3 & \text{if } i \equiv 2 \bmod 4, \text{ and} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$$

Then each tile covers a total of either 0 or 16, depending on its placement. In particular, it always covers a multiple of 16. However, a $(2m + 1) \times (16n + 8)$ rectangle covers a total that is congruent to 8 modulo 16, and so does a $(4m + 2) \times (8n + 4)$ rectangle. $\square$

2.13. Remark. Proposition 2.12 uses a single numbering to show that both types of rectangles cannot be tiled. In general, one may need several different numberings to show that several regions cannot be tiled.

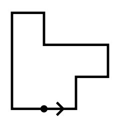
2.14. Remark. It is not hard to show that we can translate a square by 4 units, and then it is straightforward to calculate that $H(\mathcal{T}) \cong \mathbb{Z}^5 \times (\mathbb{Z}/4\mathbb{Z})$.

3. Boundary Words

In this section, we describe the boundary word method of Conway and Lagarias. This is a non-abelian analogue of tile homology, although that may not be immediately clear from the construction!

We must make an important assumption here. Our prototiles must be simply connected. We also assume that they have connected interior, although this condition can be relaxed in some cases. Such a tile has a *boundary word*, obtained by starting at a lattice point on the boundary, and traversing the boundary. For definiteness, we will always traverse in the counterclockwise direction. A unit step in the positive x [respectively, y] direction is transcribed as an x [respectively, y]. A step in the negative x [respectively, y] direction is transcribed as x^{-1} [respectively, y^{-1}].

3.1. Example. Consider the following hexomino with the indicated base point.



$x\,y\,x\,y\,x^{-2}y\,x^{-1}y^{-3}x$

Figure 3.2. Example of boundary word.

Its boundary word is $xyxyx^{-2}yx^{-1}y^{-3}x$. We note that the boundary word depends upon

(1) the choice of base point, and
(2) the particular orientation of the tile.

With regard to (1), a different base point gives rise to a conjugate boundary word. Condition (2) forces us to use translation-only tiles; therefore if we want to allow rotations and/or reflections, we must explicitly include each valid orientation in our protoset. This is actually advantageous, because we may use this to restrict the orientations that occur, for example, to forbid reflections of a tile. We will do this in one example below.

The significance of boundary words is the relationship between the boundary word of a region and the boundary words of the tiles that occur in a tiling. This is given by the following (note that our statement is slightly stronger than that given by Conway and Lagarias).

3.3. Theorem. (Conway-Lagarias) *Suppose that the simply connected region R is tiled by $T_1, T_2, \ldots, T_n$, one copy of each. Then a boundary word of R can be written as*

$$w_R = \widetilde{w}_1\widetilde{w}_2 \cdots \widetilde{w}_n$$

where $\widetilde{w}_i$ is conjugate to a boundary word of T_i, this being an identity in the free group on the generators x and y.

Proof. We argue by induction on n. The case $n = 1$ is trivial. So suppose that $n > 1$, and that the theorem holds for all simply connected regions tiled by fewer than n tiles. Fix a tiling of R by $T_1, T_2, \ldots, T_n$, and consider one of the tiles, T, that meets the boundary of R. Suppose it meets the boundary along $k \geq 1$ segments, some of which may be isolated points. Removing T from the region results in a new region with k components, $R_1, R_2, \ldots, R_k$, some of which may touch at a corner. We label the boundary word of each R_i as $v_i^{-1} u_i$, where u_i is the word along the part of the boundary shared with R, and v_i is along the part shared with the boundary of the tile T. Let $t_1, t_2, \ldots, t_k$ be the words along the segments where T meets the boundary of R. Then we may take for a boundary word of R the element $w_R = t_1 u_1 t_2 u_2 \cdots t_k u_k$. A boundary word for T is then $w_T = t_1 v_1 t_2 v_2 \cdots t_k v_k$. (In Figure 3.4, t_2 is the empty word.)

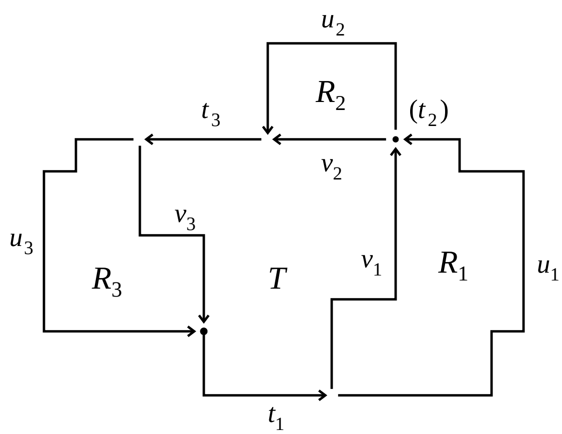


Figure 3.4. Decomposition of tiling.

Thus we have

$$w_R = w_T \widetilde{w}_{R_1} \widetilde{w}_{R_2} \cdots \widetilde{w}_{R_k} \tag{3.5}$$

where each $\widetilde{w}_{R_i} = (t_{i+1} v_{i+1} t_{i+2} v_{i+2} \cdots t_k v_k)^{-1} (v_i^{-1} u_i)(t_{i+1} v_{i+1} t_{i+2} v_{i+2} \cdots t_k v_k)$ is a conjugate of the boundary word of R_i. The induction hypothesis applies to each R_i, and each tile occurs precisely once in T and the tilings of the R_i's. Thus (3.5) implies that

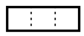
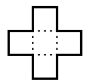
$$w_R = w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n)}$$

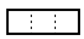
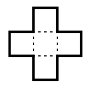
where σ is a permutation of $\{1, 2, \ldots, n\}$, and w_i is conjugate to a boundary word of T_i. It is easy to show that this implies that $w_R = \widetilde{w}_1 \widetilde{w}_2 \cdots \widetilde{w}_n$ where each $\widetilde{w}_i$ is conjugate to w_i. This completes the induction and the proof. □

An immediate consequence is the following.

3.6. Corollary. *Suppose that x and y are elements of a group G, such that the boundary word of every tile in $\mathcal{T}$ is the identity element of G. If a (simply connected) region can be tiled by $\mathcal{T}$, then its boundary word also gives the identity element of G.* □

3.7. Remark. The converse of Corollary 3.6 is false in general, even if G is taken to be the "largest" group in which the boundary words of all tiles in $\mathcal{T}$ are trivial. This is due to the non-abelian analogue of signed tilings (see Corollary 6.6 below).

3.8. Example. $\mathcal{T} = \{$  ,  $\}$, allowing all orientations. Torsten Sillke asked if these two polyominoes could tile any rectangle whose area is not a multiple of 3. The next result shows that the answer to his query is "no".

3.9. Theorem. *If* $\mathcal{T} = \{$  ,  $\}$ *tiles a rectangle, then one side is divisible by* 3.

Proof. First note that it suffices to prove that $\mathcal{T}$ cannot tile any rectangle both of whose dimensions are congruent to 1 modulo 3. For if $\mathcal{T}$ tiles a $(3m+2)\times(3n+1)$ rectangle, then two of these tilings may be juxtaposed to give a tiling of a $(6m+4)\times(3n+1)$ rectangle. Similarly, if $\mathcal{T}$ tiles a $(3m+2)\times(3n+2)$ rectangle, then it also tiles a $(6m+4)\times(6n+4)$ rectangle. Thus we need only show that $\mathcal{T}$ cannot tile any $(3m+1)\times(3n+1)$ rectangle. Let x be the 3-cycle $(1,2,3)\in S_5$, and let y be the 3-cycle $(3,4,5)$. Then we easily check that $x^3yx^{-3}y^{-1} = xy^3x^{-1}y^{-3} = xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1} = 1$, so the boundary words of all tiles are trivial. However, the boundary word of a $(3m+1)\times(3n+1)$ rectangle is $x^{3m+1}y^{3n+1}x^{-(3m+1)}y^{-(3n+1)} = (2,3,5)$, so it cannot be tiled. $\square$

3.10. Remark. A 1×1 square has a signed tiling by $\mathcal{T}$, so the tile homology technique cannot prove this result.

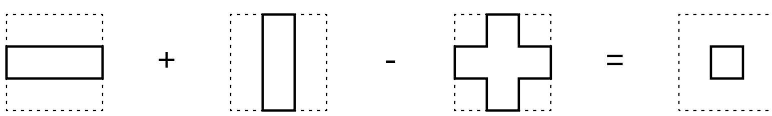


Figure 3.11. Signed tiling of a 1×1 square.

3.12. Remark. One might suspect that every rectangular tiling by $\mathcal{T}$ uses only the straight tromino. If this were the case, then the theorem would be somewhat less interesting, and a proof could be given by a checkerboard type argument. However, a 10×15 rectangle has a tiling by $\mathcal{T}$ which actually uses the X pentomino.

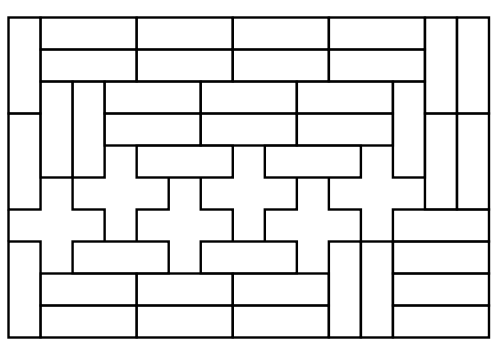


Figure 3.13. 10×15 rectangle.

3.14. Question. Is there a rectangular tiling by $\mathcal{T}$ that uses exactly 3 X pentominoes?

Theorem 3.9 shows that the number of X's in a rectangular tiling must be a multiple of 3. The tiling of Figure 3.13 has 6 X's and the following tiling has 9.

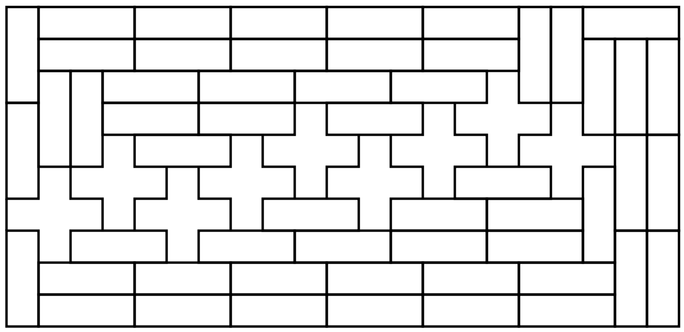


Figure 3.15. 10×21 rectangle, with 9 X pentominoes.

From the tilings in Figures 3.13 and 3.15, it is easy to construct rectangular tilings with $3n$ X's, for any $n \geq 2$.

4. Tile Homotopy Groups

We establish some notation. Let P be the free group on generators x and y. We may think of elements of P as paths on the square lattice. For a protoset $\mathcal{T}$, let $Z(\mathcal{T})$ be the smallest normal subgroup of P that contains all boundary words of tiles in $\mathcal{T}$. Corollary 3.6 strongly suggests the following definition.

4.1. Definition. The *tile path group* of $\mathcal{T}$, denoted $P(\mathcal{T})$, is the quotient group $P/Z(\mathcal{T})$.

Corollary 3.6 says that if a region R has a tiling by $\mathcal{T}$, then its boundary word is trivial in the tile path group. However, it may be difficult to work with the tile path group directly. Theorem 3.9 above uses a representation of the tile path group to show that the boundary word of a $(3m+1) \times (3n+1)$ rectangle is non-trivial.

4.2. Notation. From now on, we will use x and y to denote the free generators of P, and $\bar{x}, \bar{y}$ to denote their images in $P(\mathcal{T})$.

The tile path group turns out to be "too big" in a sense. Let $C \subseteq P$ be the subgroup of *closed words*, i.e. those words that correspond to a closed path. (For the square lattice, this is simply the commutator subgroup of P; for other lattices it won't necessarily be so.) It is clear that C is indeed a subgroup, and is normal in P. Also, since every boundary word is a closed word, we have $Z(\mathcal{T}) \subseteq C$. Now we have the insightful definition of Conway and Lagarias.

4.3. Definition. The *tile homotopy group* of $\mathcal{T}$ is the quotient $\pi(\mathcal{T}) = C/Z(\mathcal{T})$.

The relevance of this group is two-fold. Firstly, we are interested in the boundary word of a region, modulo $Z(\mathcal{T})$. Since every boundary word is closed, only elements of $\pi(\mathcal{T})$ need to be considered. Secondly, there is a strong connection between the tile homotopy group and the tile homology group, which we now examine.

Relation between Tile Homotopy and Tile Homology

To understand the tile homotopy group, we first seek an understanding of the group C of closed words. This is a subgroup of the free group P, so the following classical result of Nielsen and Schreier is relevant. See [9, Chapter 7, Section 2] for more about this. We sketch its proof, because we are interested in producing an explicit set of free generators of C.

4.4. Theorem. *Any subgroup of a free group is also free.*

Proof. (Sketch) Let G be a free group and let $H \subseteq G$ be a subgroup. Then G is the fundamental group of a bouquet of circles, X, one circle for each generator. Subgroups of G are in one-to-one correspondence with (connected) covering spaces of X. Thus H corresponds to a covering space $Y \to X$, where elements of H are exactly those closed paths on X that lift to closed paths on Y, and the fundamental group of Y is precisely H. Since Y is a graph (i.e. a 1-dimensional CW complex), the proof is finished by the following proposition.

4.5. Proposition. *The fundamental group of a graph is free.*

Proof. (Sketch) Let Γ be a graph, $T \subseteq \Gamma$ a spanning tree, and $\{e_\alpha\}$ the set of edges in the complement of T. Suppose also that the edges e_α are equipped with a favored orientation. Then one shows that $\pi_1(\Gamma)$ is free on generators $\{g_\alpha\}$ which are in bijective correspondence to the edges $\{e_\alpha\}$. The generator g_α is the class of the path defined as follows. First, traverse a path inside the tree T from the basepoint to the initial endpoint of the edge e_α, then cross the edge e_α, and finally, return to the basepoint through the tree T. It is easy to show that the g_α's generate $\pi_1(\Gamma)$; that they form a set of free generators is a consequence of Van Kampen's theorem. □

By examining the details of this proof, we can identify a free generating set for the group of closed words, C. Firstly, P is the fundamental group of a bouquet of 2 circles, X. It is easy to identify the covering space corresponding to C, this is simply the skeleton of the square lattice in the plane, call it Y. Of course, there is no canonical choice of spanning tree of Y, nor does there seem to be a "best" choice. We will use the spanning tree consisting of all horizontal edges along the x-axis, and all vertical edges. For the edges in the complement, we choose their favored orientation to be right to left, as shown in Figure 4.6.

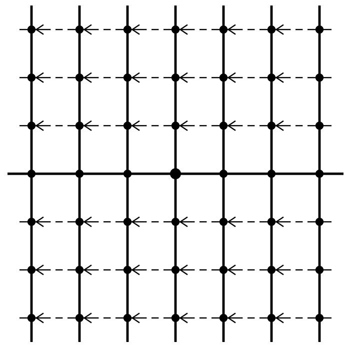


Figure 4.6. Spanning tree.

Thus we see that C is a free group on the generators

$$b_{ij} = x^{i+1}y^jx^{-1}y^{-j}x^{-i} \qquad \text{for } i \in \mathbb{Z},\ j \in \mathbb{Z} \setminus \{0\}\ .$$

However, it seems better to use a different set of free generators. We will use the following set.

4.7. Proposition. *C is a free group on the generators $c_{ij} = x^iy^jxyx^{-1}y^{-1}y^{-j}x^{-i}$, over all $i, j \in \mathbb{Z}$.*

Proof. We just saw that C is freely generated by the elements $b_{ij} = x^{i+1}y^jx^{-1}y^{-j}x^{-i}$ for $i \in \mathbb{Z}$ and $j \in \mathbb{Z} \setminus \{0\}$. Therefore we can express each c_{ij} uniquely as a word in the b_{ij}'s; the explicit expression is

$$c_{ij} = \begin{cases} b_{i1} & \text{if } j = 0, \\ b_{i,-1}^{-1} & \text{if } j = -1, \\ b_{ij}^{-1}b_{i,j+1} & \text{otherwise.} \end{cases} \tag{4.8}$$

We can also express the b_{ij}'s in terms of the c_{ij}'s; we get

$$b_{ij} = \begin{cases} c_{i0}c_{i1}c_{i2}\cdots c_{i,j-1} & \text{if } j > 0, \\ c_{i,-1}^{-1}c_{i,-2}^{-1}c_{i,-3}^{-1}\cdots c_{ij}^{-1} & \text{if } j < 0. \end{cases} \tag{4.9}$$

Therefore the c_{ij}'s generate C. Now we must show freeness. Let G be any group, and for $i, j \in \mathbb{Z}$, let g_{ij} be any element of G. We must show that there is a unique homomorphism $\varphi : C \to G$ with $\varphi(c_{ij}) = g_{ij}$ for all i, j. Expression (4.9) shows that any such φ must satisfy

$$\varphi(b_{ij}) = \begin{cases} g_{i0}g_{i1}g_{i2}\cdots g_{i,j-1} & \text{if } j > 0, \\ g_{i,-1}^{-1}g_{i,-2}^{-1}g_{i,-3}^{-1}\cdots g_{ij}^{-1} & \text{if } j < 0. \end{cases} \tag{4.10}$$

Since the b_{ij}'s are free generators, there is a unique homomorphism $\varphi : C \to G$ satisfying (4.10). Then equation (4.8) shows that indeed $\varphi(c_{ij}) = g_{ij}$ for all i, j. This shows that the c_{ij}'s are free generators, which completes the proof. $\square$

The significance of the c_{ij}'s is that c_{ij} has winding number 1 around the (i, j) cell and has winding number 0 around all other cells. Now we are in good position to understand the relation between tile homotopy and tile homology.

4.11. Theorem. (Conway-Lagarias) *The abelianization of the tile homotopy group of $\mathcal{T}$ is its tile homology group.*

Proof. We have $\pi(\mathcal{T})^{\text{ab}} = (C/Z(\mathcal{T}))^{\text{ab}} \cong C^{\text{ab}}/(\text{image of } Z(\mathcal{T}))$. As C is free on the generators c_{ij}, C^{ab} is a free abelian group on the images of these generators. The generators are in bijective correspondence with the cells of the square lattice, so we may think of C^{ab} as the free abelian group on these cells. It remains to determine the image of $Z(\mathcal{T})$ under this identification. $Z(\mathcal{T})$ is generated by all P-conjugates of boundary words of tiles in $\mathcal{T}$. A typical such generator has the form uwu^{-1}, where $u \in P$ is arbitrary, and w is a boundary word of a tile. This corresponds to a closed path (thus an element of C), so it may be written uniquely as a word in the c_{ij}'s. To understand its image in C^{ab}, we need to know the total weight with which each c_{ij} occurs. However, this weight is simply the winding number around cell (i, j), and the winding number is either 1 or 0, depending upon whether the cell occurs in the tile placement or not. Thus the image of uwu^{-1} is the element in the free abelian group on cells that corresponds to this particular tile placement. Now we see that $\pi(\mathcal{T})^{\text{ab}} \cong C^{\text{ab}}/(\text{image of } Z(\mathcal{T})) \cong A/B(\mathcal{T})$, which is the tile homology group. □

Hidden behind the scenes is a topological space, which we now bring to the forefront. Let Y be the skeleton of the square lattice, which we have seen in Figure 4.6. Note that $Y \to X$ is a normal covering map, where X is a bouquet of two circles, and the group of deck transformations is $\mathbb{Z}^2$, acting via translations of the square lattice.

From Y, we build a new space, called $Y(\mathcal{T})$, by sewing in a 2-cell into every possible tile placement. This is a covering space for $X(\mathcal{T})$, which is constructed in a similar way. Namely, we sew in the boundary of a 2-cell along the path corresponding to each boundary word of a tile in $\mathcal{T}$. (Technically, we must sew in a cell for *every possible* boundary word, where all possible base points are considered.) Then $Y(\mathcal{T}) \to X(\mathcal{T})$ is also a normal covering map, again whose group of deck transformations is $\mathbb{Z}^2$ acting via translations of the square lattice. Moreover, the restriction to Y is the covering map $Y \to X$.

The fundamental group of $X(\mathcal{T})$ is the tile path group $P(\mathcal{T})$, and the covering space $Y(\mathcal{T})$ corresponds to the subgroup $\pi(\mathcal{T}) \subseteq P(\mathcal{T})$. The first homology group of $Y(\mathcal{T})$ is the tile homology group, $H(\mathcal{T})$. Thus Theorem 4.11 can be considered as a special case of the Hurewicz Isomorphism Theorem.

5. Strategy for working with Tile path groups

We have shown above how to translate tiling problems into problems in finitely presented groups, so we might hope to be able to resolve such questions. Unfortunately, the situation is grim. The so-called *word problem*, as well as many related problems, is known to be unsolvable, which means that no algorithm can answer the question for all possible values of the input.

This is not the end of our story, for we are not trying to solve every word problem. We might hope, however optimistically, that the word problems that arise for us can be solved, whether by hook or by crook. The algorithmic unsolvability of these problems should serve to temper any optimism that we can muster.

The tile homotopy method has been successfully applied in several cases, see [2, Exercise for Experts] [4] [13] [14]. Despite these efforts, results have been found in only a handful of cases. In this section, we give a simple strategy for understanding tile homotopy groups, which allows many new cases to be handled. In view of the difficulty in working with finitely presented groups, we understand that our approach cannot be algorithmic, nor can we expect to be able to apply it in all cases. Nonetheless, we are able to use our strategy to handle numerous new cases.

The tile path group for a finite set $\mathcal{T}$ of prototiles is given by a finitely presented group. We are more interested in the tile homotopy group, which is a subgroup of infinite index. The infiniteness of this index is unfortunate, in light of the following well-known result.

5.1. Proposition. *If G is a finitely generated [respectively, finitely presented] group, and $H \subseteq G$ is a subgroup of finite index, then H is also finitely generated [respectively, finitely presented].* □

The usual proof uses covering space theory, similar to the determination of the group, C, of closed paths above. Moreover, in the finitely presented case, a presentation of H can be computed explicitly. We will do this later, with the help of the computer software package GAP [5]. There is plenty of interesting combinatorial group theory involved in this, but it is well understood, so it is not our place to discuss it here.

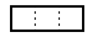
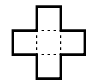
If the index $(G : H)$ is not finite, then H can fail to be finitely generated. A typical example exhibiting this behavior is the case $C \subseteq P$ that we saw earlier.

In general, the tile homotopy group will not be finitely generated. However, in some special cases, it will be. The method of demonstrating this is a non-abelian analogue of the technique for showing finite generation of the tile homology group, as in Examples 2.5 and 2.7. In order to achieve this, we need to find some relations in the tile path group.

5.2. Theorem. *Suppose that $\bar{x}^m$ and $\bar{y}^n$ are central in $P(\mathcal{T})$, for some positive m and n. Then the natural map $P(\mathcal{T}) \twoheadrightarrow \widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^m, \bar{y}^n \rangle$ induces an isomorphism of $\pi(\mathcal{T})$ onto its image, $\widetilde{\pi}(\mathcal{T})$. Moreover, $\widetilde{\pi}(\mathcal{T})$ has index mn inside $\widetilde{P}(\mathcal{T})$ and it is generated by the images of the elements $\bar{c}_{ij} = \bar{x}^i \bar{y}^j \bar{x} \bar{y} \bar{x}^{-1} \bar{y}^{-1} \bar{y}^{-j} \bar{x}^{-i}$ for $0 \leq i < m$ and $0 \leq j < n$.*

Proof. Note that $\pi(\mathcal{T})$ is normal in $P(\mathcal{T})$, with quotient $P(\mathcal{T})/\pi(\mathcal{T}) \cong P/C \cong \mathbb{Z}^2$. This quotient is the group of translations of the grid, so $\bar{x}$ and $\bar{y}$ map to rightward and upward translation by 1 unit each. Let $N = \langle \bar{x}^m, \bar{y}^n \rangle \subseteq P(\mathcal{T})$, which, by hypothesis, is central in $P(\mathcal{T})$. Now N maps injectively to $P(\mathcal{T})/\pi(\mathcal{T})$, whence N and $\pi(\mathcal{T})$ intersect trivially. Thus $\pi(\mathcal{T})$ maps injectively to $P(\mathcal{T})/N = \widetilde{P}(\mathcal{T})$. This proves the first statement. Next, note that $\widetilde{P}(\mathcal{T})/\widetilde{\pi}(\mathcal{T}) \cong P(\mathcal{T})/N\pi(\mathcal{T}) \cong \mathbb{Z}^2/\langle \text{image of } \bar{x}^m, \bar{y}^n \rangle \cong (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$. This shows that the index $(\widetilde{P}(\mathcal{T}) : \widetilde{\pi}(\mathcal{T})) = mn$, as claimed. Finally, we recall that $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij}$ over all $i, j \in \mathbb{Z}$. Since $\bar{x}^m$ is central in $P(\mathcal{T})$, we see that $\bar{c}_{ij} = \bar{c}_{i+m,j}$, and $\bar{c}_{ij} = \bar{c}_{i,j+n}$, because $\bar{y}^n$ is also central. The last statement is then clear. $\square$

Theorem 5.2 is an important tool for calculating tile homotopy groups. We revisit an example (3.8) we had seen earlier.

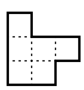
5.3. Theorem. *The tile homotopy group of* $\mathcal{T} = \{$  ,  $\}$ *has order* 120*, and it is a central extension of* A_5 *by* $\mathbb{Z}/2\mathbb{Z}$*.*

Proof. The tile path group has the presentation

$$P(\mathcal{T}) = \langle x, y \mid x^3yx^{-3}y^{-1}, xy^3x^{-1}y^{-3}, xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1} \rangle.$$

The relators show that $\bar{x}^3$ and $\bar{y}^3$ are central in $P(\mathcal{T})$. Let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^3, \bar{y}^3 \rangle = \langle x, y \mid x^3, y^3, xyxyx^{-1}yx^{-1}y^{-1}x^{-1}y^{-1}xy^{-1} \rangle$. Then the projection $P(\mathcal{T}) \twoheadrightarrow \widetilde{P}(\mathcal{T})$ induces an isomorphism of $\pi(\mathcal{T})$ onto its image $\widetilde{\pi}(\mathcal{T})$, which has index 9 in the finitely presented group $\widetilde{P}(\mathcal{T})$. Thus we can compute a presentation of $\widetilde{\pi}(\mathcal{T})$. In this particular instance, we have an even better situation, because the group $\widetilde{P}(\mathcal{T})$ turns out to be finite, and therefore $\widetilde{\pi}(\mathcal{T})$ is also finite. In fact, GAP quickly tells us that $|\widetilde{P}(\mathcal{T})| = 1080$, so that $\pi(\mathcal{T})$ has order 120, and its structure can be completely determined. $\square$

The utility of Theorem 5.2 depends on the ability to find relations in the tile path group. It is known that this cannot be done algorithmically, but in some cases, it is easy to find the necessary relations. In Theorem 5.3, it was trivial to find them. In the next theorem, the relations are not quite as obvious.

5.4. Theorem. *Let* $\mathcal{T} = \{$  $\}$*, with all orientations allowed.*

(a) *The tile homotopy group* $\pi(\mathcal{T})$ *is solvable. Its derived series is* $\pi(\mathcal{T}) = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 = \{1\}$*, with quotients* $G_0/G_1 = \pi(\mathcal{T})^{\mathrm{ab}} = H(\mathcal{T}) \cong \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})$*,* $G_1/G_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ *and* $G_2/G_3 = G_2 \cong \mathbb{Z}/2\mathbb{Z}$*. Moreover, these isomorphisms can be given explicitly.*
(b) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then (at least) one of* m *or* n *is a multiple of* 4*.*
(c) *A* 2×3 *rectangle has a signed tiling by* $\mathcal{T}$*.*

Proof. We first claim that $\bar{x}^{12}$ and $\bar{y}^{12}$ are central in $P(\mathcal{T})$. Consider the two tilings shown in Figure 5.5.

Figure 5.5. Two small tilings.

The first shows that $\bar{x}^3$ commutes with $\bar{y}^2\bar{x}\bar{y}^2$, and the second shows that $\bar{x}^4$ commutes with $\bar{y}^2\bar{x}\bar{y}$. Therefore, $\bar{x}^{12}$ commutes with both $\bar{y}^2\bar{x}\bar{y}^2$ and $\bar{y}^2\bar{x}\bar{y}$, and thus also with $\bar{y}$. Hence

$\bar{x}^{12}$ is central in $P(\mathcal{T})$, and similarly, $\bar{y}^{12}$ is also central. Let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^{12}, \bar{y}^{12}\rangle$. Theorem 5.2 shows that $\pi(\mathcal{T})$ maps isomorphically onto its image in $\widetilde{P}(\mathcal{T})$, with finite index. Now we can compute a presentation of $\pi(\mathcal{T})$, using GAP. We obtain

$$G_0 = \pi(\mathcal{T}) \cong \langle z_1, z_2 \mid z_2 z_1 z_2 z_1 z_2 z_1^{-2}, z_1 z_2^2 z_1^{-1} z_2^{-2}\rangle,$$

where the generators are $z_1 = \bar{x}^{-1}\bar{y}\bar{x}\bar{y}^{-1}$ and $z_2 = \bar{y}^2\bar{x}\bar{y}^{-2}\bar{x}^{-1}$. From this, we find that

$$H(\mathcal{T}) = \pi(\mathcal{T})^{\mathrm{ab}} \cong \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z}).$$

There are two different ways we can make this isomorphism explicit. Firstly, we can express the image of each c_{ij} in terms of z_1 and z_2, and then use the explicit presentation of $\pi(\mathcal{T})$ above. However, it is much easier to compute $H(\mathcal{T})$ directly. We have

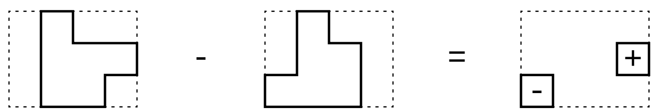


Figure 5.6. Translating a square 3 units to the right and 1 unit up.

which shows how we can translate a square 3 units to the right and 1 unit up. By considering all 8 orientations of this relation, we find that we can translate a square by 1 diagonal unit. Now it is easy to see that $H(\mathcal{T}) \cong \mathbb{Z}\times(\mathbb{Z}/3\mathbb{Z})$ is given by $[R] \mapsto (b-r, (b+r) \bmod 3)$, where the region R contains b black squares and r red squares in the usual checkerboard coloring.

Next we compute the commutator subgroup $G_1 = [G_0, G_0]$. We cannot do this directly, because it has infinite index in G_0. However, we can utilize the same technique as in Theorem 5.2 above. The first relator implies that $z_1^3 = (z_2 z_1)^3$. Therefore, z_1^3 commutes with $z_2 z_1$, and hence is central in G_0. Now let $N = \langle z_1^3\rangle \subseteq G_0$. We see that N maps injectively to $G_0^{\mathrm{ab}} = G_0/G_1$, so that G_1 maps injectively to $G_0/N = \langle z_1, z_2 \mid z_1^3, z_2 z_1 z_2 z_1 z_2 z_1^{-2}, z_1 z_2^2 z_1^{-1} z_2^{-2}\rangle$. Moreover, its image has index 9 in G_0/N. Now GAP can compute a presentation of G_1; it tells us that

$$G_1 \cong \langle a_1, a_2 \mid a_1^2 a_2^2, a_1 a_2 a_1 a_2^{-1}\rangle,$$

where $a_1 = z_2 z_1 z_2^{-1} z_1^{-1}$ and $a_2 = z_2 z_1^{-1} z_2^{-1} z_1$. Also, G_1 is easily seen to be a finite group (quaternion of order 8). Thus the rest of (a) can be readily verified.

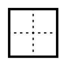
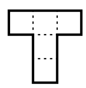
(b) It suffices to show that $\mathcal{T}$ cannot tile any $(4m+2)\times(4n+2)$ rectangle. Having already completely determined the structure of the tile homotopy group, we content ourselves with a representation proof. Define $\varphi : P(\mathcal{T}) \to S_{32}$ by

$$\begin{aligned}\varphi(\bar{x}) = {} & (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)\\ & (21,22,23,24)(25,26,27,28)(29,30,31,32)\\ \varphi(\bar{y}) = {} & (1,4,32,20)(2,12,7,17)(3,24,23,11)(5,16,15,21)(6,13,27,18)\\ & (8,22,10,28)(9,19,29,25)(14,31,30,26)\end{aligned}$$

It is straightforward to check that this indeed gives a homomorphism; one only needs to verify that the boundary words of all eight orientations are in the kernel of φ. We also note that $\varphi(\bar{x}^{4m+2}\bar{y}^{4n+2}\bar{x}^{-(4m+2)}\bar{y}^{-(4n+2)})$ is non-trivial, so a $(4m+2)\times(4n+2)$ rectangle cannot be tiled by $\mathcal{T}$.

(c) This follows from the explicit isomorphism $H(\mathcal{T}) \cong \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})$ given above. Also, an explicit signed tiling is easy to give, based upon Figure 5.6 above. $\square$

We remark that these computations depend upon the correctness of the computer program. If a proof of non-tileability relies on this computation, it may be advantageous to give a *certificate of proof*, namely a homomorphism $P(\mathcal{T}) \to G$ to a group in which we can compute easily. Having done that, the representation proof can be easily verified, and is less susceptible to error.

5.7. Theorem. *Let* $\mathcal{T} = \{$  ,  $\}$, *with all orientations allowed.*

(a) *The tile homotopy group of* $\mathcal{T}$ *has order* 32 *and is a central extension of* $(\mathbb{Z}/2\mathbb{Z})^4$ *by* $\mathbb{Z}/2\mathbb{Z}$.
(b) *The tile homology group,* $H(\mathcal{T}) \cong (\mathbb{Z}/2\mathbb{Z})^4$, *and a specific isomorphism is given as follows. Suppose that the region* R *covers* X_0 *[respectively,* X_1, X_2*] cells with* x*-coordinate congruent to* $0 \bmod 3$ *[respectively,* $1 \bmod 3$, $2 \bmod 3$*]. Also, suppose that* R *covers* Y_0 *[respectively,* Y_1, Y_2*] cells with* y*-coordinate* $\equiv 0 \bmod 3$ *[respectively,* $1 \bmod 3$, $2 \bmod 3$*]. Then a specific isomorphism* $H(\mathcal{T}) \cong (\mathbb{Z}/2\mathbb{Z})^4$ *is given by*

$$[R] \mapsto ((X_0 + X_1) \bmod 2, (X_1 + X_2) \bmod 2, (Y_0 + Y_1) \bmod 2, (Y_1 + Y_2) \bmod 2).$$

(c) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is even.*
(d) *A* 3×3 *square has a signed tiling by* $\mathcal{T}$.

Proof. (a) We first claim that $\bar{x}^6$ is central in $P(\mathcal{T})$. Consider the two tilings below.

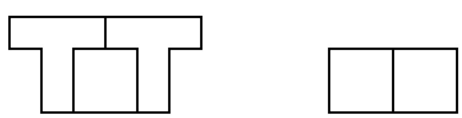


Figure 5.8. Two small tilings.

They show that

$$\bar{y}^{-2}\bar{x}^4\bar{y}^2\bar{x}\bar{y}\bar{x}^{-6}\bar{y}^{-1}\bar{x} = 1 \quad \text{and} \quad \bar{y}^{-2}\bar{x}^4\bar{y}^2\bar{x}^{-4} = 1,$$

so that $\bar{x}\bar{y}\bar{x}^{-6}\bar{y}^{-1}\bar{x} = \bar{x}^{-4}$. This shows that $\bar{x}^6$ commutes with $\bar{y}$ and therefore is central. Similarly, $\bar{y}^6$ is central in $P(\mathcal{T})$. Now let $\widetilde{P}(\mathcal{T}) = P(\mathcal{T})/\langle \bar{x}^6, \bar{y}^6 \rangle$. Theorem 5.2 shows that $\pi(\mathcal{T})$ maps isomorphically onto its image in $\widetilde{P}(\mathcal{T})$, and it has index 36. We can now compute

$$\pi(\mathcal{T}) \cong \langle z_1, z_2, z_3, z_4 \mid z_1^2, z_2^2, z_3^2, z_4^4, (z_1z_2)^2z_4^2, (z_1z_3)^2z_4^2, (z_2z_3)^2z_4^2, (z_1z_4)^2, (z_2z_4)^2, (z_3z_4)^2 \rangle$$

where $z_1 = \bar{y}\bar{x}\bar{y}^{-1}\bar{x}^{-1}$, $z_2 = \bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{x}$, $z_3 = \bar{x}\bar{y}\bar{x}\bar{y}^{-1}\bar{x}^{-2}$ and $z_4 = \bar{y}^2\bar{x}\bar{y}^{-2}\bar{x}^{-1}$. We can easily check that this group is finite, and its structure can be completely determined. In fact, the relators make it clear that z_4^2 is central, has order 2, generates the commutator subgroup, and the quotient $\pi(\mathcal{T})/\langle z_4^2\rangle$ is an elementary abelian 2-group of rank 4.

(b) We show how we can translate a square by 3 units.

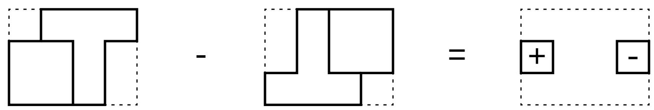


Figure 5.9. Translating a square by 3 units.

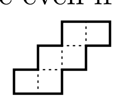
Now a straightforward computation, similar to Examples 2.5 and 2.7, shows that $H(\mathcal{T}) \cong (\mathbb{Z}/2\mathbb{Z})^4$, and the isomorphism is as claimed.

(c) We must show that $\mathcal{T}$ cannot tile a $(2m+1)\times(2n+1)$ rectangle, so it suffices to show that $\mathcal{T}$ cannot tile a $(6m+3)\times(6n+3)$ rectangle. We use a representation proof. Define a homomorphism $\varphi : P(\mathcal{T}) \to S_{48}$ by

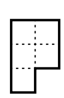
$$\begin{aligned}
\varphi(\bar{x}) = {} & (1,13,11,12,10,16)(2,41,34,25,38,31)(3,42,35,26,39,32)(4,40,36,27,37,33) \\
& (5,20,46,6,23,43)(7,19,17,48,22,14)(8,28,9,29,45,30)(15,44,21,18,47,24) \\
\varphi(\bar{y}) = {} & (1,27,30,12,10,23,29,18,11,8,28,31)(2,13,14,46,4,44,43,45,3,25,15,40) \\
& (5,20,24,35,9,21,41,34,33,19,16,36)(6,26,39,22,47,42,38,17,48,32,37,7)
\end{aligned}$$

It is straightforward to verify that this indeed defines a homomorphism. Furthermore, we easily check that $\varphi(\bar{x}^{6m+3}\bar{y}^{6n+3}\bar{x}^{-(6m+3)}\bar{y}^{-(6n+3)})$ is non-trivial, so a $(6m+3)\times(6n+3)$ cannot be tiled by $\mathcal{T}$.

(d) This follows from the isomorphism $H(\mathcal{T}) \cong (\mathbb{Z}/2\mathbb{Z})^4$ given in part (b). Also, it is easy to give an explicit one, based upon Figure 5.9. $\square$

5.10. Remark. The tilings in Figure 5.8 and the argument involved essentially amount to "untiling" two square tetrominoes from the left figure. This is the non-abelian analogue of a signed tiling. Since the boundary word of the 1×6 rectangle is trivial in $P(\mathcal{T})$, Theorem 5.7 remains true even if this rectangle is included in the protoset $\mathcal{T}$. We can also show that the hexomino  has such a "generalized tiling" by $\mathcal{T}$, so this shape may also be included in $\mathcal{T}$, and Theorem 5.7 remains valid.

We give one more example.

5.11. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where rotations are allowed, but reflections are prohibited.*

(a) *The tile homotopy group,* $\pi(\mathcal{T})$, *is a central extension of* $\mathbb{Z}^4$ *by* $\mathbb{Z}/2\mathbb{Z}$. *In particular, it is solvable.*

(b) *The tile homology group is* $H(\mathcal{T}) \cong \mathbb{Z}^4$*, and an explicit isomorphism is given as follows. Suppose that the region* R *covers* n_0 *[respectively,* n_1, n_2, n_3, n_4*] cells* (i, j) *with* $2i + j \equiv 0 \bmod 5$ *[respectively,* $1 \bmod 5,\ 2 \bmod 5,\ 3 \bmod 5,\ 4 \bmod 5$*]. Then an explicit isomorphism* $H(\mathcal{T}) \xrightarrow{\cong} \mathbb{Z}^4$ *is given by* $[R] \mapsto (n_1 - n_0, n_2 - n_0, n_3 - n_0, n_4 - n_0)$*.*
(c) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is even.*
(d) *A* 1×5 *rectangle has a signed tiling by* $\mathcal{T}$*.*

Proof. (a) Note that $\mathcal{T}$ tiles a 2×5 rectangle, which implies that $\bar{x}^2$ commutes with $\bar{y}^5$. Similarly, $\bar{x}^5$ commutes with $\bar{y}^2$. Therefore, $\bar{x}^{10}$ commutes with $\bar{y}$ and thus is central in $P(\mathcal{T})$. In the same way, $\bar{y}^{10}$ is also central in $P(\mathcal{T})$, so we can compute a presentation of $\pi(\mathcal{T})$, using Theorem 5.2. We obtain a presentation for $\pi(\mathcal{T})$ with 5 generators: $z_1 = \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, $z_2 = \bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{x}$, $z_3 = \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y}$, $z_4 = \bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$ and $w = z_1 z_2 z_1^{-1} z_2^{-1}$. The relations are $w^2 = 1$, $w z_i = z_i w$ for $1 \le i \le 4$, and $z_i z_j z_i^{-1} z_j^{-1} = w$ for $1 \le i < j \le 4$. The relations show that w is central in $\pi(\mathcal{T})$ and that the quotient $\pi(\mathcal{T})/\langle w \rangle$ is isomorphic to $\mathbb{Z}^4$. Furthermore, w has order 2, and it generates the commutator subgroup of $\pi(\mathcal{T})$. This proves (a).

(b) Note that we have

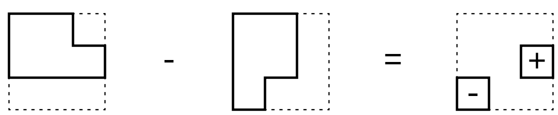


Figure 5.12. Translating a square 2 units to the right and 1 unit up.

so that $\bar{a}_{ij} = \bar{a}_{i+2,j+1}$ in $H(\mathcal{T})$. Similarly, we have $\bar{a}_{ij} = \bar{a}_{i-1,j+2}$, so $H(\mathcal{T})$ is generated by $\bar{a}_{00}, \bar{a}_{10}, \bar{a}_{20}, \bar{a}_{30}$ and $\bar{a}_{40}$. Furthermore, the relations collapse into a single relation: $\bar{a}_{00} + \bar{a}_{10} + \bar{a}_{20} + \bar{a}_{30} + \bar{a}_{40} = 0$. Thus $H(\mathcal{T}) \cong \mathbb{Z}^4$, and the isomorphism is as claimed.

(c) It suffices to show that $\mathcal{T}$ cannot tile a $(10m + 5) \times (10n + 5)$ rectangle. We use a representation proof. Define a homomorphism $\varphi : P(\mathcal{T}) \to S_{64}$ by

$$\begin{aligned}
\varphi(\bar{x}) = {} & (1,2,4,47,16,27,41,54,56,9)(3,6,12,11,34,50,62,61,49,58) \\
& (5,10,19,32,24,36,31,37,42,55)(7,14,23,28,43,57,52,40,38,46)(8,59) \\
& (13,21,35,51,20,15,25,17,18,30)(22,33,48,60,64,26,39,53,63,44)(29,45) \\
\varphi(\bar{y}) = {} & (2,3,5,9,17,28,42,12,14,22)(4,7,13,6,20)(8,25,37,11,33) \\
& (10,18,29,44,58)(15,24,30,46,57,63,62,48,54,47)(16,26,38,50,61) \\
& (19,31,39,45,21,34,49,51,59,64)(27,40)(32,36,52,35,41)(43,56,60,55,53)
\end{aligned}$$

As usual, it is straightforward to verify that φ indeed defines a homomorphism, and that $\varphi(\bar{x}^{10m+5}\bar{y}^{10n+5}\bar{x}^{-(10m+5)}\bar{y}^{-(10n+5)})$ is non-trivial.

(d) This follows from the explicit isomorphism $H(\mathcal{T}) \xrightarrow{\cong} \mathbb{Z}^4$ given above. Alternatively, it is easy to give a signed tiling, based upon Figure 5.12. $\square$

6. Criteria for $\pi(\mathcal{T})$ to be abelian

In many cases that we have examined, the tile homotopy group turns out to be abelian. In such cases, the tile homotopy group gives no further information than the tile homology group, which is generally more accessible. We give here two general criteria which imply that $\pi(\mathcal{T})$ is abelian.

6.1. Theorem. *Suppose that the set of prototiles $\mathcal{T}$ is rotationally invariant.*
(a) *If $\bar{x}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in $P(\mathcal{T})$, then $\pi(\mathcal{T})$ is cyclic, and its order is the greatest common divisor of the sizes of tiles in $\mathcal{T}$. If d is this greatest common divisor, then a specific isomorphism $\pi(\mathcal{T})\xrightarrow{\cong}\mathbb{Z}/d\mathbb{Z}$ is given by $[\gamma]\mapsto N \bmod d$, where the loop γ encloses N squares, counting multiplicity.*
(b) *If $\bar{x}\bar{y}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in $P(\mathcal{T})$, then $\pi(\mathcal{T})$ is abelian. Let $H\subseteq\mathbb{Z}^2$ be the subgroup generated by all elements of the form (b,r) and (r,b), where there is a tile in $\mathcal{T}$ with b black squares and r red squares. Then $\pi(\mathcal{T})\cong\mathbb{Z}^2/H$, and a specific isomorphism is given by $[\gamma]\mapsto(B,R) \bmod H$, where the loop γ encloses B black squares and R red squares, counting multiplicity.*

Proof. (a) A $90°$ clockwise rotation corresponds to mapping x and y to y^{-1} and x respectively. Since $\mathcal{T}$ is invariant under this rotation, this map induces an automorphism of $P(\mathcal{T})$. Thus $\bar{y}^{-1}$ commutes with $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$, and therefore also with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ is central in $P(\mathcal{T})$. We have seen that $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij}=\bar{x}^i\bar{y}^j\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}\bar{y}^{-j}\bar{x}^{-i}$, and our commutativity relations show that these are all equal to $\bar{c}=\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Thus $\pi(\mathcal{T})$ is generated by a single element, $\bar{c}$, and therefore is cyclic.

Let $w\in C$ be the boundary word of a tile in $\mathcal{T}$, which imposes a relation upon $P(\mathcal{T})$. Then w can be written uniquely as a word in the elements c_{ij}. The total weight in an individual c_{ij} is the winding number around square (i,j), which is either 1 or 0, according to whether or not that square is in the tile. Thus the total weight in all the c_{ij}'s is the size of the tile. Therefore, w imposes the relation $\bar{c}^n=1$ on $\pi(\mathcal{T})$, where n is the size of the tile. The remainder of the statement is now clear.

(b) Since $\bar{x}\bar{y}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, so does $\bar{x}^{-1}\bar{y}^{-1}$. A $90°$ clockwise rotation shows that $\bar{y}^{-1}\bar{x}$ commutes with $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}^{-1}$, and conjugating by $\bar{y}$ shows that $\bar{x}\bar{y}^{-1}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now we see that both $\bar{x}^2=(\bar{x}\bar{y}^{-1})(\bar{x}^{-1}\bar{y}^{-1})^{-1}$ and $\bar{y}^2=(\bar{x}^{-1}\bar{y}^{-1})^{-1}(\bar{x}\bar{y}^{-1})^{-1}$ also commute with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Next, $\pi(\mathcal{T})$ is generated by the elements $\bar{c}_{ij}$. Our commutativity relations show that $\bar{c}_{ij}=\bar{c}_{00}$ if $i+j$ is even, while $\bar{c}_{ij}=\bar{c}_{10}$ if $i+j$ is odd. Moreover, these two elements commute with each other, because $\bar{c}_{00}=\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$, and $\bar{c}_{10}=(\bar{x}^2)(\bar{x}\bar{y}^{-1})^{-1}(\bar{x}\bar{y})^{-1}$.

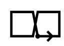
Let $w\in C$ be the boundary word of a tile in $\mathcal{T}$, which may be written uniquely as a word in the elements c_{ij}. The total weight in those c_{ij}'s with $i+j$ even [respectively, odd] is the number of black [respectively, red] squares in this placement of the tile. Thus w imposes the relation $\bar{c}_{00}^b\bar{c}_{10}^r=1$ on $\pi(\mathcal{T})$, and the relation $\bar{c}_{00}^r\bar{c}_{10}^b=1$ comes from the boundary word xwx^{-1}. The statement now follows. $\square$

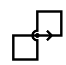
It may be useful to reformulate Theorem 6.1 in a different way. We will consider the following self-intersecting closed paths to depict "generalized tiles" that have boundary words $xyx^{-1}y^{-1}x^{-1}yxy^{-1}$ and $xyx^{-1}y^{-2}x^{-1}yx$ respectively.

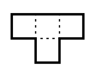
Figure 6.2. Generalized tiles.

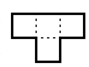
Now Theorem 6.1 may be rephrased as follows.

6.3. Theorem. *Suppose that rotations are allowed in our protosets.*

(a) *The tile homotopy group of $\mathcal{T} = \{$  $\}$ is isomorphic to $\mathbb{Z}$, and a specific isomorphism is given by $[\gamma] \mapsto N$, where the loop γ encloses N squares, counting multiplicity.*

(b) *The tile homotopy group of $\mathcal{T} = \{$  $\}$ is isomorphic to $\mathbb{Z}^2$, and a specific isomorphism is given by $[\gamma] \mapsto (B, R)$, where the loop γ encloses B black squares and R red squares, counting multiplicity.* □

Conway and Lagarias mention the protoset $\mathcal{T} = \{$  $\}$, with all orientations allowed. They remark that Walkup [17] has shown that if an $m \times n$ rectangle can be tiled by $\mathcal{T}$, then both m and n are multiples of 4. They also note that a rectangle has a signed tiling by $\mathcal{T}$ if and only if its area is a multiple of 8. They implicitly ask what the relationship between Walkup's proof and the tile homotopy method is. Theorem 6.1 above allows us to compute the tile homotopy group of $\mathcal{T}$.

6.4. Proposition. *The tile homotopy group of $\mathcal{T} = \{$  $\}$ is $\mathbb{Z}/8\mathbb{Z}$. A specific isomorphism is given by $[\gamma] \mapsto (B + 5R) \bmod 8$, where the loop γ encloses B black squares and R red squares, counting multiplicity.*

Proof. The boundary words of the orientations

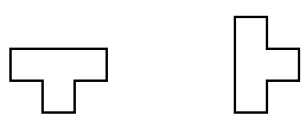
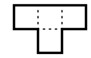


Figure 6.5. Two orientations of the T tetromino.

give the relations $\bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}^{-3}\bar{y}^{-1}\bar{x} = 1 = \bar{y}^{-1}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}^{-1}\bar{y}\bar{x}^{-1}\bar{y}^{-2}$ in $P(\mathcal{T})$. Therefore, $\bar{x}^{-2}\bar{y}^{-1}\bar{x} = \bar{y}\bar{x}^{-1}\bar{y}^{-2}$, which is equivalent to $\bar{x}\bar{y}$ commuting with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$. Now part (b) of Theorem 6.1 shows that $\pi(\mathcal{T}) \cong \mathbb{Z}^2/\langle(1,3),(3,1)\rangle \cong \mathbb{Z}/8\mathbb{Z}$, and the specific isomorphism is as claimed. □

6.6. Corollary. *The boundary word of a rectangle is trivial in* $\pi(\{$  $\})$ *if and only if its area is divisible by* 8. $\square$

This shows that Walkup's proof is unrelated to tile homotopy; his proof relies on subtle geometric restrictions that are not detected by the tile homotopy group.

Another example that exhibits a similar phenomenon in a more obvious manner is the following.

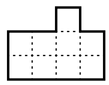
6.7. Example. Let $\mathcal{T} = \{$  $\}$, with all orientations allowed. Comparing the two orientations

Figure 6.8. Two orientations of a tile.

shows that $\bar{x}$ commutes with $\bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}$ in the tile path group. Then Theorem 6.1(a) shows that $\pi(\mathcal{T}) \cong \mathbb{Z}/9\mathbb{Z}$. This means that the tile homotopy group only detects area, modulo 9.

On the other hand, we can easily show that if $\mathcal{T}$ tiles a rectangle, then both sides must be even. Consider the ways that a tile can touch the edge of a rectangle.

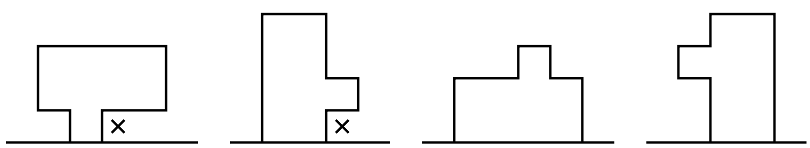
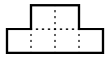


Figure 6.9. Tiles along an edge of a rectangle.

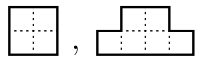
We see that the first two possibilities cannot occur, so each tile that touches the edge does so along an even length. Therefore, each edge of the rectangle has even length. In fact, it is not much harder to show that if $\mathcal{T}$ tiles an $m \times n$ rectangle, then both m and n are multiples of 6. A straightforward argument shows that every tiling of a quadrant by $\mathcal{T}$ is a union of 6×6 squares, which implies the result.

7. Appendix: Further examples

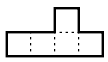
Here we give some more tiling restrictions we have found using the tile homotopy technique. In each case, there are signed tilings that show that the result cannot be obtained by tile homology methods, and there are tilings that show that the result is non-vacuous. Further details will be published elsewhere.

7.1. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then either* m *or* n *is a multiple of* 4.
(b) *A* 1×6 *rectangle has a signed tiling by* $\mathcal{T}$.

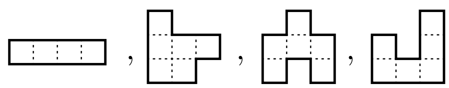
7.2. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is a multiple of* 4.
(b) *A* 1×6 *rectangle has a signed tiling by* $\mathcal{T}$.

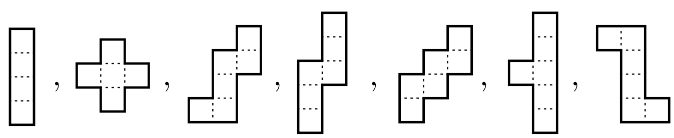
7.3. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where rotations are permitted, but reflections are not.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is even.*
(b) *A* 1×5 *rectangle has a signed tiling by* $\mathcal{T}$.

7.4. Remark. It is easy to show that if $\mathcal{T}$ tiles a rectangle, then both sides are multiples of 5. Also, Yuri Aksyonov [1] has given a clever geometric proof that one side must be a multiple of 10.

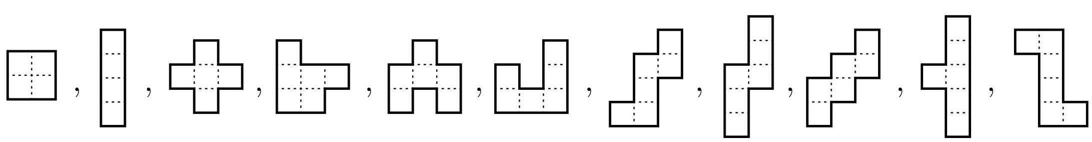
7.5. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 4.
(b) *A* 1×2 *rectangle has a signed tiling by* $\mathcal{T}$.

7.6. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

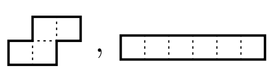
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 4.
(b) *A* 1×2 *rectangle has a signed tiling by* $\mathcal{T}$.

7.7. Theorem. *Let*

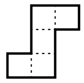
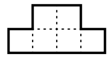
$\mathcal{T} = \{$  $\}$,

where all orientations are allowed.

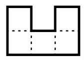
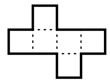
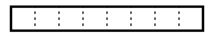
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is a multiple of* 4.
(b) *A* 1×2 *rectangle has a signed tiling by* $\mathcal{T}$.

7.8. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

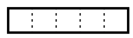
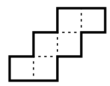
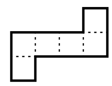
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 6.
(b) *A* 2×2 *square has a signed tiling by* $\mathcal{T}$.

7.9. Theorem. *Let* $\mathcal{T} = \{$  ,  $\}$, *where all orientations are allowed.*

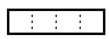
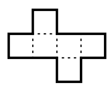
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then either* m *is a multiple of* 3 *or* n *is a multiple of* 6.
(b) *A* 1×1 *square has a signed tiling by* $\mathcal{T}$.

7.10. Theorem. *Let* $\mathcal{T} = \{$  ,  ,  $\}$, *where all orientations are allowed.*

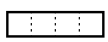
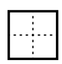
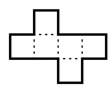
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 8.
(b) *A* 1×1 *square has a signed tiling by* $\mathcal{T}$.

7.11. Theorem. *Let* $\mathcal{T} = \{$  ,  ,  $\}$, *where all orientations are allowed.*

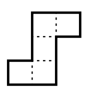
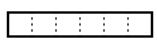
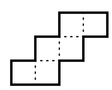
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 5.
(b) *A* 1×1 *square has a signed tiling by* $\mathcal{T}$.

7.12. Theorem. *Let* $\mathcal{T} = \{$  ,  $\}$, *where all orientations are allowed.*

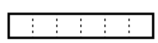
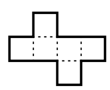
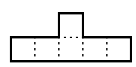
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 4.
(b) *A* 1×2 *rectangle has a signed tiling by* $\mathcal{T}$.

7.13. Theorem. *Let* $\mathcal{T} = \{$  ,  ,  $\}$, *where all orientations are allowed.*

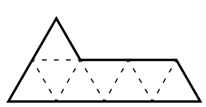
(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then* mn *is a multiple of* 4.
(b) *A* 1×2 *rectangle has a signed tiling by* $\mathcal{T}$.

7.14. Theorem. *Let* $\mathcal{T} = \{$  ,  ,  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 6.
(b) *A* 1×1 *square has a signed tiling by* $\mathcal{T}$.

7.15. Theorem. *Let* $\mathcal{T} = \{$  ,  ,  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles an* $m \times n$ *rectangle, then one of* m *or* n *is a multiple of* 6.
(b) *A* 2×3 *rectangle has a signed tiling by* $\mathcal{T}$.

7.16. Theorem. *Let* $\mathcal{T} = \{$  $\}$, *where all orientations are allowed.*

(a) *If* $\mathcal{T}$ *tiles a triangle of side* n, *then* n *is a multiple of* 8.
(b) *A triangle of side* 4 *has a signed tiling by* $\mathcal{T}$.

7.17. Remark. That $\mathcal{T}$ tiles any triangle is quite interesting. Karl Scherer [15, 2.6 D] has found a tiling of a side 32 triangle by $\mathcal{T}$.

Acknowledgment

I thank Torsten Sillke for some interesting discussions.

References

1. Yuri Aksyonov, *E-mail communication to Torsten Sillke* (March, 1999), `http://www.mathematik.uni-bielefeld.de/~sillke/PENTA/qu5-y-right`.
2. Elwyn R. Berlekamp, John H. Conway and Richard K. Guy, *Winning Ways for your mathematical plays*, vol. 2, Academic Press, London, 1982. MR 84h:90091b
3. Max Black, *Critical Thinking*, Prentice-Hall, New York, 1946.
4. J.H. Conway and J.C. Lagarias, *Tiling with Polyominoes and Combinatorial Group Theory*, Journal of Combinatorial Theory, Series A **53** (1990), 183–208. MR 91a:05030
5. The GAP group, *GAP – Groups, Algorithms and Programming, version 4.3* (2002), `http://www.gap-system.org`.
6. Michael R. Garey and David S. Johnson, *Computers and Intractability*, Freeman, San Francisco, 1979. MR 80g:68056
7. S.W. Golomb, *Checker Boards and Polyominoes*, American Mathematical Monthly **61** (1954), 675–682. MR 16,664a
8. S.W. Golomb, *Covering a Rectangle with L-tetrominoes, Problem E1543*, American Mathematical Monthly **69** (1962), no. 9, 920.
   Solution by D.A. Klarner, American Mathematical Monthly **70** (1963), no. 7, 760–761.
9. Marshall Hall, Jr., *The Theory of Groups*, Chelsea, New York, N.Y., 1976. MR 54 #2765
10. David A. Klarner, *Packing a Rectangle with Congruent N-ominoes*, Journal of Combinatorial Theory **7** (1969), 107–115. MR 40 #1894
11. Harry Langman, *Play Mathematics*, Hafner, New York, 1962.
12. Cristopher Moore and John Michael Robson, *Hard Tiling Problems with Simple Tiles*, Discrete & Computational Geometry **26** (2001), no. 4, 573–590. MR 2002h:52029
13. Igor Pak, *Ribbon Tile Invariants*, Transactions of the American Mathematical Society **352** (2000), no. 12, 5525–5561. MR 2001g:05039
14. James Propp, *A Pedestrian Approach to a Method of Conway, or, A Tale of Two Cities*, Mathematics Magazine **70** (1997), no. 5, 327–340. MR 98m:52031
15. Karl Scherer, *A Puzzling Journey To The Reptiles And Related Animals*, Privately published, Auckland, 1987, `http://karlscherer.com/Mybooks/bkrintro.html`.
16. Alexander Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, Chichester, 1986. MR 88m:90090
17. D.W. Walkup, *Covering a Rectangle with T-tetrominoes*, American Mathematical Monthly **72** (1965), 986–988. MR 32 #1582

Michael Reid
Department of Mathematics
University of Central Florida
Orlando, FL 32816
U.S.A.
`reid@cflmath.com`