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DISTANCES FORBIDDEN BY SOME TWO-COLORING OF $\mathbb{Q}^2$

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0. Introduction

In 1973, D. R. Woodall [3] showed that there is a 2-coloring of $\mathbb{Q}^2$ (the rational points in the plane) that forbids the distance 1 (i.e. no two points of the same color are at distance 1 apart). Recently, this result was strengthened by P. D. Johnson, Jr. [2], who showed there is a 2-coloring of $\mathbb{Q}^2$ that simultaneously forbids all Euclidean distances of the form $\sqrt{p/q}$ with p and q odd positive integers. Johnson's 2-coloring obviously forbids certain other distances (e.g. $\sqrt{6}$) because no two rational points in the plane are precisely that far apart. We show in Section 1 that Johnson's coloring is optimal: for any distance d not covered by either of these two cases, no 2-coloring of $\mathbb{Q}^2$ forbids both of the distances 1 and d (see Theorem 1.5). This settles Problem 4 of [2]. In Section 2, we give a similar solution of Problems 2 and 3 of [2].

1. Two-colorings of $\mathbb{Q}^2$

Definition. Let $D \subseteq \mathbb{R}^+$. A 2-coloring of $\mathbb{Q}^2$ *forbids* the distances D if for all $x, y \in \mathbb{Q}^2$, with $d(x, y) \in D$, the two points x and y have different colors.

Definition. For any $D \subseteq \mathbb{R}^+$, we say there is an *odd D-cycle* (in $\mathbb{Q}^2$) if there exist $s_1, s_2, \ldots, s_n \in \mathbb{Q}^2$ such that $s_1 + s_2 + \cdots + s_n = 0$ (under componentwise addition), $||s_i|| \in D$ for each i, and n is odd. (Not having an odd D-cycle means a certain subset of $\mathbb{Q}^2$ is "weakly two-free" in the language of [1, 2].)

The following lemma follows from the fact that a graph is bipartite iff it contains no odd cycle.

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Lemma 1.1. (cf. [1]) *Let $D \subseteq \mathbb{R}^+$. There is no 2-coloring of $\mathbb{Q}^2$ that forbids D iff there is an odd D-cycle.*

Lemma 1.2. (cf. [2, Lemma 1]) *Let $d \in \mathbb{R}^+$ and set $d' = kd$, where k is any odd positive integer. Then there is an odd $\{1, d\}$-cycle iff there is an odd $\{1, d'\}$-cycle.*

Proof. ($\Rightarrow$) Suppose $u_1 + u_2 + \cdots + u_l + s_1 + s_2 + \cdots + s_m = 0$, with $l + m$ odd, and where $||u_i|| = 1$ for each i, and $||s_j|| = d$ for each j. Setting $s'_j = k \cdot s_j$, we have

$$k \cdot u_1 + k \cdot u_2 + \cdots + k \cdot u_l + s'_1 + s'_2 + \cdots + s'_m = 0,$$

and $kl + m$ is odd.

($\Leftarrow$) Suppose $u_1 + u_2 + \cdots + u_l + s'_1 + s'_2 + \cdots + s'_m = 0$. Setting $s_j = \frac{1}{k} \cdot s'_j$, we have

$$u_1 + u_2 + \cdots + u_l + k \cdot s_1 + k \cdot s_2 + \cdots + k \cdot s_m = 0,$$

and $l + km$ is odd. □

Proposition 1.3. *Let $p, q \in \mathbb{Z}^+$, with p even and q odd, and assume $d = \sqrt{p/q}$ occurs as a distance between rational points in the plane. Then there is an odd $\{1, d\}$-cycle.*

Proof. By Lemma 1.2 it suffices to show there is an odd $\{1, qd\}$-cycle. Since d occurs as a distance, there are rational numbers α and β with $\alpha^2 + \beta^2 = pq$. Any integer that is a sum of two rational squares is a sum of two integer squares, so we may assume that α and β are non-negative integers. Now $\alpha \cdot (1, 0) + \beta \cdot (0, 1) + (-\alpha, -\beta) = 0$, and $||(-\alpha, -\beta)|| = \sqrt{pq} = qd$. Since $\alpha^2 + \beta^2 = pq$ is even, α and β have the same parity, so $\alpha + \beta + 1$ is odd. □

Lemma 1.4. *Let $D \subseteq \mathbb{R}^+$ and $d_0 \in \mathbb{R}^+$. Assume d_0 occurs as a distance between two rational points in the plane. Then there is an odd D-cycle iff there is an odd $(d_0 \cdot D)$-cycle (where $d_0 \cdot D = \{d_0 d \mid d \in D\}$).*

Proof. By assumption there are $\alpha, \beta \in \mathbb{Q}$ with $\sqrt{\alpha^2 + \beta^2} = d_0$. Then $\left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ is a linear transformation of the plane that preserves the set of rational points and expands all distances precisely by a factor of d_0. The lemma follows. □

Theorem 1.5. *Let $d \in \mathbb{R}^+$. Assume there are $x, y \in \mathbb{Q}^2$ with $d(x, y) = d$, and that d can be written in the form $d = \sqrt{p/q}$ with $p, q \in \mathbb{Z}^+$ and $p + q$ odd. Then no 2-coloring of $\mathbb{Q}^2$ forbids both of the distances 1 and d.*

Proof. By Lemma 1.1, we need only show there is an odd $\{1, d\}$-cycle in $\mathbb{Q}^2$. By Lemma 1.4, there is an odd $\{1, d\}$-cycle iff there is an odd $\{1/d, 1\}$-cycle. So, replacing d by $1/d$ if necessary, we may assume $d = \sqrt{p/q}$ where p is even and q is odd. Then Proposition 1.3 applies. □

Corollary 1.6. *Let $D \subseteq \mathbb{R}^+$. No 2-coloring of $\mathbb{Q}^2$ forbids the distances D iff there are $d_1, d_2 \in D$ such that:*

(1) *Each of d_1 and d_2 occurs as a distance between rational points in the plane; and*
(2) *There exist $p, q \in \mathbb{Z}^+$ such that $d_1/d_2 = \sqrt{p/q}$ and $p + q$ is odd.*

2. Two-colorings of A_n and $\mathbb{Q}^3$

Let $A_n = \{(a_1/b_1, a_2/b_2, \ldots, a_n/b_n) \mid a_i, b_i \in \mathbb{Z}$ and each b_i is odd $\}$. We now prove the natural analogue of Theorem 1.5 for A_n and for $\mathbb{Q}^3$. This settles problems 2 and 3 of [2].

Proposition 2.1. *Let $p, q \in \mathbb{Z}^+$, with p even and q odd. Set $d = \sqrt{p/q}$ and assume there are $x, y \in A_n$ with $d(x, y) = d$. Then no 2-coloring of A_n forbids both of the distances 1 and d.*

Proof. (cf. proof of Proposition 1.3) We need only show there is an odd $\{1, d\}$-cycle in A_n (see Lemma 1.1). Write $d^2 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$, where each α_i is rational with odd denominator. We may rescale in the manner of Lemma 1.2 to assume each α_i is an integer, so d^2 is an even integer. Then

$$\alpha_1 \cdot (1, 0, \ldots, 0) + \alpha_2 \cdot (0, 1, \ldots, 0) + \cdots + \alpha_n \cdot (0, 0, \ldots, 1) + (-\alpha_1, -\alpha_2, \ldots, -\alpha_n) = 0,$$

and $\alpha_1 + \alpha_2 + \cdots + \alpha_n + 1$ is odd. □

Proposition 2.2. *Let $d \in \mathbb{R}^+$. Assume there are $x, y \in \mathbb{Q}^3$ with $d(x, y) = d$, and that d can be written in the form $d = \sqrt{p/q}$ with $p, q \in \mathbb{Z}^+$ and $p + q$ odd. Then no 2-coloring of $\mathbb{Q}^3$ forbids both of the distances 1 and d.*

Proof. We need only show there is an odd $\{1, d\}$-cycle in $\mathbb{Q}^3$ (see Lemma 1.1). Write $p/q = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ (with each $\alpha_i \in \mathbb{Q}$).

Case 1. p is even and q is odd. It follows that $\alpha_1, \alpha_2, \alpha_3$ have odd denominators (when written in lowest terms). Now apply the proof of Proposition 2.1.

Case 2. p is odd and q is even. At least one of $\alpha_1, \alpha_2, \alpha_3$ must have even denominator (when written in lowest terms). Assume it is $\alpha_1 = p_1/q_1$. Note

$$p_1 \cdot (1, 0, 0) + \frac{q_1}{2} \cdot (-\alpha_1, \alpha_2, \alpha_3) + \frac{q_1}{2} \cdot (-\alpha_1, -\alpha_2, -\alpha_3) = 0,$$

and $p_1 + q_1$ is odd. □

Question. Is the analogue of Lemma 1.4 (and, hence, also the analogue of Corollary 1.6) true for A_n and $\mathbb{Q}^3$?

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